Algorithms of Scientific Computing

Exercise 1

In the last worksheet we showed that the $a_k$ and $b_k$ can be computed by

$$c_k = \frac{1}{12} \sum_{l=0}^{11} X_l e^{-i2\pi kl/12},$$  \hspace{1cm} \text{(1)}

i.e. by a DFT.

Use the idea of the Fast Fourier Transformation, to reduce this DFT of length 12 to the computation of some DFTs of length 6 or 3, respectively.

Use the fact that all $X_l \in \mathbb{R}$.

Draw a diagram, that shows the needed computation steps or write an appropriate program (for example in Maple).

Extra-Exercise for Interested People:

Try to use Edson’s algorithm or the mentioned ideas for the Fast Real Fourier Transformation for this problem.

Exercise 2: DFT of Mirrored data

Assume a dataset $f_n$, $n = 0, \ldots, N - 1$. What is the difference of the Fourier coefficients for this dataset and the “mirrored” dataset $\tilde{f}_n := f_{N-n}$?

Exercise 3: Two-dimensional Cosine-Transformation

The JPEG-method computes the coefficients $\tilde{F}_{kl}$ from the image data $f_{nm}$ using the following formula

$$\tilde{F}_{kl} = \frac{1}{N \cdot M} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f_{nm} \cos \left( \frac{\pi k (n + \frac{1}{2})}{N} \right) \cos \left( \frac{\pi l (m + \frac{1}{2})}{M} \right)$$
Assume you have a procedure that can compute all coefficients \( \tilde{G}_k, k = 0, \ldots, N - 1 \) efficiently, according to the formula

\[
\tilde{G}_k = \frac{1}{N} \sum_{n=0}^{N-1} f_n \cos \left( \frac{\pi k (n + \frac{1}{2})}{N} \right).
\]

How can you compute the coefficients \( \tilde{F}_{kl} \) using this procedure?

**Exercise 4: Discrete Cosine Transformation**

We start with a dataset \( f_{-N+1}, \ldots, f_N \), which fulfills the following symmetry constraint:

\[
f_{-n} = f_n \quad \text{for } n = 1, \ldots, N - 1
\]

a) Show that the corresponding Fourier coefficients

\[
F_k = \frac{1}{2N} \sum_{n=-N+1}^{N} f_n \omega_n^{-kn}
\]

are real values only and can be written as:

\[
F_k = \frac{1}{N} \left( \frac{1}{2} f_0 + \sum_{n=1}^{N-1} f_n \cos \left( \frac{\pi nk}{N} \right) + \frac{1}{2} f_N \cos(\pi k) \right).
\]

b) Show that the \( F_k \) is symmetric too.