Exercise 1: Hierarchization in Higher Dimensions

In this exercise we will implement the multi-recursive algorithm for hierarchization of a multi-dimensional regular sparse grid. The `SparseGrid` class in the Python source file `sparsegrid.py` provides a code skeleton in which some parts are missing.

(i) Fill the gaps in `__init__`, `hierarchizeMainAxisRecursively` and `hierarchizeRecursively`. For the specifications look at the functions’ document strings and follow the instructions in the comments.

Hint: The basic algorithm is very similar to `insertGridPointsRecursively`.

(ii) Fill in the function body of `computeVolume`.

Exercise 2: The Combination Technique – A Different View on Sparse Grids

Dealing with hierarchical bases often turns out to be sophisticated. On this worksheet we will therefore see how the so-called combination technique finds a sparse grid interpolant, that approximates a function on a number of full grids, each consisting only of a “relatively small” number of grid points.

Let $u_\ell$ ($\ell \in \mathbb{N}^2$) for a $u : [0,1]^2 \rightarrow \mathbb{R}$ the interpolant in $V_\ell$ (interpolating piecewise bilinearly at the inner grid points, at the boundary $u$ is assumed to be zero again).

(i) $V_\ell$ can be decomposed into a set of subspaces $W_\ell$. Accordingly, the interpolant $u_\ell \in V_\ell$ can be written as a sum of $w_\ell \in W_\ell$.

Spot the grid associated with $u_{(3,2)}$ in the right part of Figure 1. Identify those subspaces in the left part that are needed to reconstruct $u_{(3,2)}$.

(ii) Use the result from (i) to rewrite

$$\sum_{\|\ell\|_1=n+1} u_\ell, \quad n \in \mathbb{N}$$
Figure 1: The two parts in the picture show the grid points and supports associated with interpolants $w_L$ (left) and $u_L$ (right) up to level 4 for the 2d case.

for the two-dimensional case as a weighted sum of $w_L$.

**Hint:** Look at the subspace scheme in Figure 1 and count the occurrences of each subspace in the sum. What do you notice when comparing $w_L$ with common level $n = |\|1 + \dim - 1$?

(iii) In the final step use the previous results to give a representation of the sparse grid interpolant

$$u^D_n := \sum_{|\|1 \leq n+1} w_L$$

as a weighted sum of $u_L$. Again, count the occurrences of the $w_L$.

(iv) Assume you are talking to a person who knows how to approximate the volume $F_2(u)$ through the trapezoidal rule (in 2d) with respect to $u_L$. Give instructions on how to write a program that implements a sparse grid approximation of $F_2(u)$. Remember Archimedes quadrature.

(v) Compare this method with Archimedes quadrature — what are the (dis-)advantages?