Algorithms of Scientific Computing
(Algorithmen des Wissenschaftlichen Rechnens)
The Wavelet Scaling Function, Haar Wavelets

The wavelet families we look at (e.g., Haar wavelets) are constructed around a multiresolution analysis, a nested sequence $V_n$ of function spaces some of which properties are

$$V_j \subset V_{j+1}, \quad j \in \mathbb{Z} \quad (1)$$

$$\bigcap_{j=-\infty}^{\infty} V_j = \{0\} \quad (2)$$

$$f(t) \in V_l \iff f(2^{-l}t) \in V_0 \quad (3)$$

$$V_l = V_{l-1} \oplus W_{l-1}$$
$$= V_{l-2} \oplus W_{l-2} \oplus W_{l-1}$$
$$= V_0 \oplus W_0 \oplus W_1 \oplus \cdots \oplus W_{l-1}, \quad (4)$$

with orthogonal functions $f \in V_j$ and $g \in W_j$, i.e. $\langle f, g \rangle = 0$.

The theory of multiresolution analysis further states the existence of a unique function $\phi$ which satisfies a so-called dilation equation of the form

$$\phi(t) = \sum_{k \in \mathbb{Z}} c_k \cdot \phi(2t - k). \quad (5)$$

and which helps us define orthonormal nodal bases $\{\phi_{l,k}\}$ for the $V_l$ with

$$\phi_{l,k}(t) = \phi(2^l t - k)$$
$$\text{span} \{\phi_{l,k}\} = V_l, \quad \langle \phi_{l,k}, \phi_{l,m}\rangle = \delta_{k,m} \quad k, m \in \mathbb{Z}. \quad (6)$$

The function $\phi$ is called father wavelet or scaling function, and together with a mother wavelet $\psi$ it defines the wavelet family. It is not necessary to know a specific formula for $\phi$, the dilation equation (5) with its coefficients $c_k$ together with the theory of multiresolution analysis provide enough information to derive the mother wavelet $\psi$ as well as orthonormal wavelet bases $\{\psi_{l,m}\}$ for the $W_l$ with

$$\psi_{l,k}(t) = \psi(2^l t - k)$$
$$\text{span} \{\psi_{l,k}\} = W_l, \quad \langle \psi_{l,k}, \psi_{l,m}\rangle = \delta_{k,m} \quad k, m \in \mathbb{Z}. \quad (7)$$
1 Cranking The Machine

Typically the scaling function $\phi$ is not known explicitly, and sometimes a closed-form analytic formula does not even exist. However, for continuous $\phi$ we can approximate the function to arbitrarily high precision using the “Cascade Algorithm”, a fixed-point method for functions.

In this exercise we want to implement this algorithm by iterating over the expression

$$F(\gamma)(t) = \sum_k c_k \cdot \gamma(2t - k)$$

in order to find the fixed point $\gamma$ of $F$.

Our starting point $\gamma_0$ will be the hat function

$$\gamma_0(t) = \begin{cases} 
1 + t & \text{for } -1 \leq t \leq 0 \\
1 - t & \text{for } 0 < t \leq 1 \\
0 & \text{else}
\end{cases}$$

(i) Over the interval $[-1; 3]$ plot the approximations of the scaling function $\phi$ for the Haar wavelet family obtained in the first 7 iterations of the cascade algorithm. Do so by plugging the refinements coefficients $c_k$, $k = 0, 1$ in (10) into (8) resp. (5).

$$c_0 = c_1 = 1$$

(ii) Over the interval $[-1; 3]$ plot the approximations of the scaling function $\phi$ for the Daubechies wavelet family obtained in the first 7 iterations of the cascade algorithm. Do so by plugging the refinements coefficients $c_k$, $k = 0, \ldots, 3$ in (11) into (8) resp. (5).

$$c_0 = \frac{1 + \sqrt{3}}{4} \quad c_1 = \frac{3 + \sqrt{3}}{4} \quad c_2 = \frac{3 - \sqrt{3}}{4} \quad c_3 = \frac{1 - \sqrt{3}}{4}$$

2 The Haar Wavelet Basis

In this exercise we want to compute the 1-d wavelet transform for the Haar wavelet family and apply it to a signal vector $\vec{s}$ of length $m = 2^n$. The transform can be implemented very efficiently as a “pyramidal algorithm” taking $O(m)$ steps. For educational purpose we focus on the $O(m^2)$ matrix-based algorithm.

(i) Write a function that constructs the transformation matrix $M$ consisting of the basis vectors $\psi_{l,k}$, $l \leq n$, $0 \leq k \leq 2^n - 1$.

(ii) Use Python’s package numpy.linalg to invert the matrix.

(iii) Use the program to compute the transform $M\vec{s} = \vec{d}$ as well as the reconstructed signal $M^{-1}\vec{d} = \vec{s}'$ of the vector

$$\vec{s} = [1, 2, 3, -1, 1, -4, -2, 4]^T$$

(iv) Verify the program’s output tracing the steps by hand.