Algorithms of Scientific Computing

Space-Filling Curves and their Applications in Scientific Computing (3)

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**Approximating Polygons of the Hilbert Curve**

**Idea:** Connect start and end point of iterate on each subcell.

**Definition:**

The straight connection of the $4^n + 1$ points

\[ h(0), h(1 \cdot 4^{-n}), h(2 \cdot 4^{-n}), \ldots, h((4^n - 1) \cdot 4^{-n}), h(1) \]

is called the *n-th approximating polygon of the Hilbert curve*.
Properties of the Approximating Polygon

- the approximating Polygon connects the corners of the recursively divided subsquares
- the connected corners are start and end points of the space-filling curve within each subsquare
  ⇒ assists in the construction of space-filling curves
- approximating polygons are constructed by recursive repetition of a so-called Leitmotiv
  ⇒ similarity to Koch and other fractal curves
- the sequence of corresponding functions \( p_n(t) \) converges uniformly towards \( h \)
  ⇒ additional proof of continuity of the Hilbert curve
Example: Koch Curve
Construction of the Peano Curve

Recursive Construction:

- divide quadratic domain into 9 subsquares
- construct Peano curve for each subsquare
- join the partial curves to build a higher level curve
Arithmetic Formulation of the Peano Function

$t$ given in “nonal” system, $t = 0.9.n_1 n_2 n_3 n_4 \ldots$, then

$$p(0.9.n_1 n_2 n_3 n_4 \ldots) = P_{n_1} \circ P_{n_2} \circ P_{n_3} \circ P_{n_4} \circ \cdots \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with the operators

$$P_2 \left( \begin{array}{c} x \\ y \end{array} \right) = \begin{pmatrix} \frac{1}{3} x + 0 \\ \frac{1}{3} y + \frac{2}{3} \end{pmatrix} \quad P_3 \left( \begin{array}{c} x \\ y \end{array} \right) = \begin{pmatrix} \frac{1}{3} x + \frac{1}{3} \\ -\frac{1}{3} y + 1 \end{pmatrix} \quad P_8 \left( \begin{array}{c} x \\ y \end{array} \right) = \begin{pmatrix} \frac{1}{3} x + \frac{2}{3} \\ \frac{1}{3} y + \frac{2}{3} \end{pmatrix}$$

$$P_1 \left( \begin{array}{c} x \\ y \end{array} \right) = \begin{pmatrix} -\frac{1}{3} x + \frac{1}{3} \\ \frac{1}{3} y + \frac{1}{3} \end{pmatrix} \quad P_4 \left( \begin{array}{c} x \\ y \end{array} \right) = \begin{pmatrix} -\frac{1}{3} x + \frac{2}{3} \\ -\frac{1}{3} y + \frac{2}{3} \end{pmatrix} \quad P_7 \left( \begin{array}{c} x \\ y \end{array} \right) = \begin{pmatrix} -\frac{1}{3} x + 1 \\ \frac{1}{3} y + \frac{1}{3} \end{pmatrix}$$

$$P_0 \left( \begin{array}{c} x \\ y \end{array} \right) = \begin{pmatrix} \frac{1}{3} x + 0 \\ \frac{1}{3} y + 0 \end{pmatrix} \quad P_5 \left( \begin{array}{c} x \\ y \end{array} \right) = \begin{pmatrix} \frac{1}{3} x + \frac{1}{3} \\ -\frac{1}{3} y + \frac{1}{3} \end{pmatrix} \quad P_6 \left( \begin{array}{c} x \\ y \end{array} \right) = \begin{pmatrix} \frac{1}{3} x + \frac{2}{3} \\ \frac{1}{3} y \end{array} \right)$$
Approximating Polygons of the Peano Curve

Definition:

The straight connection between the $9^n + 1$ points

$$p(0), p(1 \cdot 9^{-n}), p(2 \cdot 9^{-n}), \ldots, p((9^n - 1) \cdot 9^{-n}), p(1)$$

is called $n$-th approximating polygon of the Peano curve
Peano’s Representation of the Peano Curve

**Definition:** (Peano curve, original construction by G. Peano)

- each $t \in \mathcal{I} := [0, 1]$ has a ternary representation
  
  $$t = (0_3.t_1 t_2 t_3 t_4 \ldots)$$

- define the mapping $p: \mathcal{I} \to \mathcal{Q} := [0, 1] \times [0, 1]$ as

  $$p(t) := \left(\begin{array}{c}
  0_3.t_1 \ k^{t_2}(t_3) \ k^{t_2+t_4}(t_5) \ldots \\
  0_3.k^{t_1}(t_2) \ k^{t_1+t_3}(t_4) \ldots
  \end{array}\right)$$

  where $k(t_i) := 2 - t_i$ for $t_i = 0, 1, 2$ and $k^j$ is the $j$-times concatenation of the function $k$
Peano’s Representation of the Peano Curve (2)

Still to prove:

- $p$ is independent of the ternary representation
- the Peano curve $p : I \to Q$ defines a space-filling curve.

Comments:

- the direction of “switchback” can be both vertical (see definition), horizontal, or mixed;
- actually, **272 different Peano curves** of the switchback type can be constructed using the same principles;
  For comparison: there are only two different 2D Hilbert curves
- in addition: 2 Peano-Meander curves
How Long are Approximating Polygons?

Example: Hilbert curve

- polygon results from recursive repetition of the Leitmotiv
- every recursion step **doubles** the length of the polygon in each subsquare

\[ \Rightarrow \text{length of the } n\text{-th polygon is } 2^n \to \infty \text{ for } n \to \infty. \]

Corollaries:

- the “length” of the Hilbert curve is not well defined
- instead, we can give an “area” of the Hilbert curve
  \[ (1, \text{the area of the unit square}) \]

\[ \Rightarrow \textbf{Question: what’s the dimension of a Hilbert curve?} \]
Fractal Dimension of Curves

Measuring the length of a curve:

- approx. the curve by a polygon with faces of length $\epsilon$
  $\Rightarrow$ gives a measured length $L(\epsilon)$.
  (cmp. approximating polygons of a space-filling curve)

- in case of recursive repeat of a Leitmotiv:
  replace each units of length $r$ by a polygon of length $q$, then
  $$L\left(\frac{\epsilon}{r}\right) = \frac{q}{r}L(\epsilon), \quad L(1) := \lambda$$

- we obtain for the length $L(\epsilon)$:
  $$L(\epsilon) = \lambda \epsilon^{1-D}, \quad \text{where} \quad D = \log_r q = \frac{\log q}{\log r}$$
Length of a recursively defined curve computed as

\[ L(\epsilon) = \lambda \epsilon^{1-D}, \quad \text{mit} \quad D = \log_r q = \frac{\log q}{\log r} \]

⇒ \( D \) is the \textbf{fractal dimension} of the curve

⇒ \( \lambda \) is the length w.r.t. that dimension

Gives “well defined” dimension:

- in all other “dimensions”, the length is 0 or \( \infty \)!
- the fractal dimension of the 2D Hilbert curve is 2, similar for the Peano curve

⇒ \textbf{Hausdorff dimension}
How Long is the Coastline of Britain?

Compare, e.g., Mandelbrot: The Fractal Geometry of Nature
Test: Length of Fractal Curves

![Graph showing the length of different fractal curves as a function of epsilon. The curves represent different fractals: circle, koch, gosper, and britain. The x-axis represents epsilon, ranging from 0.1 to 30, and the y-axis represents length, ranging from 0 to 400. Each curve is marked with data points and labeled accordingly.]
Exercise: What is the Area of a Fractal Curve?

Koch curve as example:

→ refine green area and compute its limit value . . .