Exercise 1: DFT and Least Square Approximation

Suppose we are given \(N\) (odd number) data pairs that we denote \((x_n, f_n)\), where \(n = -(N-1)/2: (N-1)/2\). The \(x_n\) are real and are assumed to be equally spaced points on the interval \([-A/2, A/2]\), so \(x_n = n\delta x\) where \(\delta x = A/N\). The \(f_n\) may be complex valued. We want to find an approximation to the data using the \(N\)-trigonometric polynomial \(\phi_N\), given by

\[
\phi_N(x) = \sum_{k=-(N-1)/2}^{(N-1)/2} \alpha_k e^{2\pi i k x / A},
\]

the function \(\phi_N\) is called a trigonometric polynomial because it is a polynomial in the quantity \(e^{2\pi i x / A}\).

Use the least square criterion, where we minimize the discrete least squares error

\[
E = \sum_{n=-(N-1)/2}^{(N-1)/2} |f_n - \phi_N(x_n)|^2,
\]

to find \(N\) coefficients \(\alpha_{-(N-1)/2}, \ldots, \alpha_{(N-1)/2}\).

Hint: use the expression for the partial derivative of the error \(E\):

\[
\frac{\partial E}{\partial \alpha_k} = \sum_{n=-(N-1)/2}^{(N-1)/2} \left[ e^{-i2\pi nk/N} \left( f_n - \sum_{p=-(N-1)/2}^{(N-1)/2} \alpha_p e^{2\pi ip n / N} \right) \right],
\]

and set these derivatives to 0.

Discrete Sine Transformation
optional exercises (not explained during tutorial)

Exercise 2: Fast Discrete Sine Transformation

Formulate the butterfly scheme for equation

\[
F_k = \frac{1}{2N} \sum_{n=-(N+1)}^{N} f_n \omega_{2N}^{-kn},
\]

where dataset \(f_{-N+1}, \ldots, f_N \in \mathbb{R}\) fulfills the following symmetry constraint:

\(f_n = -f_n\)
Divide the dataset $f_n$ of length $2N$ into a dataset $g_n := f_{2n}$, containing all values with an even index, and a dataset $h_n := f_{2n-1}$, with all values with odd index. Which symmetries can be found in $g_n$ and $h_n$? Of which kind (Cosine/Sine Transformation, DFT with real data) are the according DFTs of length $N$? Which symmetries can be found if the dataset $f_n$ fulfills the following symmetry constraint:

$$ f_{-n} = -f_{n+1}. $$