Algorithms of Scientific Computing
Hierarchical Methods and Sparse Grids
– 1D Hierarchical Basis –

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Archimedes’ Quadrature

Compute an approximation of $F_1 := \int_0^1 4 \cdot x \cdot (1 - x) \, dx = \frac{2}{3}$
Archimedes’ Quadrature (2)

- Integrating $4x(1 - x)$, we have to consider several quantities.
- Ordered by (recursive) level $t$:

<table>
<thead>
<tr>
<th>Level-depth</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>...</th>
<th>$t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mesh-width $h$</td>
<td>$1/2$</td>
<td>$1/4$</td>
<td>$1/8$</td>
<td>$1/16$</td>
<td>...</td>
<td>$2^{-t}$</td>
</tr>
<tr>
<td># triangles</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>...</td>
<td>$\frac{1}{2}2^t$</td>
</tr>
<tr>
<td>surplus $\nu$</td>
<td>1</td>
<td>$1/4$</td>
<td>$1/16$</td>
<td>$1/64$</td>
<td>...</td>
<td>$4 \cdot 2^{-2t}$</td>
</tr>
<tr>
<td>Area of triangle $D_1$</td>
<td>$1/2$</td>
<td>$1/16$</td>
<td>$1/128$</td>
<td>$1/1024$</td>
<td>...</td>
<td>$4 \cdot 2^{-3t}$</td>
</tr>
<tr>
<td>Sum (current $t$)</td>
<td>$1/2$</td>
<td>$1/8$</td>
<td>$1/32$</td>
<td>$1/128$</td>
<td>...</td>
<td>$2 \cdot 2^{-2t}$</td>
</tr>
<tr>
<td>Sum ($\leq t$)</td>
<td>$1/2$</td>
<td>$5/8$</td>
<td>$21/32$</td>
<td>$85/128$</td>
<td>...</td>
<td>$\frac{2}{3} \left(1 - 2^{-2t}\right)$</td>
</tr>
<tr>
<td>Error</td>
<td>$1/6$</td>
<td>$1/24$</td>
<td>$1/96$</td>
<td>$1/384$</td>
<td>...</td>
<td>$\frac{2}{3}2^{-2t}$</td>
</tr>
</tbody>
</table>
Approximation of Functions

- To analyze Archimedes’ quadrature rule, we consider functions
- We need a representation of the (approximating) function $u(x)$ which we are integrating:
  - $u$ as linear combination of ansatz functions $\phi_i$:
    $$u(x) = \sum_{i=1}^{n} \alpha_i \cdot \phi_i(x)$$

- Integrating $u(x)$:
  $$\int_{a}^{b} u(x) \, dx = \sum_{i}^{n} \alpha_i \int_{a}^{b} \phi_i(x) \, dx,$$

- Weighted sum of $\alpha_i$
- Remember: Newton-Cotes formulas are weighted sum of function evaluations
Composite Trapezoidal Rule: Function

**Interpolant**
- Continuous, piecewise linear function
- Represent $u$ in nodal point (hat) basis

- Coefficients $\alpha_i$ are function values at grid points
- Basis functions have area $h$ ($h/2$ at boundaries)
Piecewise Linear Functions

Ansatz space and basis functions

- Only consider $u : [0, 1] \rightarrow \mathbb{R}$
- Consider discretization level $n \in \mathbb{N}$
- Mesh-width $h_n = 2^{-n}$
- Grid points $x_{n,i} = i \cdot h_n$
- Define “mother of all hat functions”
  \[
  \phi(x) := \max\{1 - |x|, 0\}
  \]

⇒ Basis functions

- Nodal point basis $\Phi_n := \{\phi_{n,i}, 0 \leq i \leq 2^n\}$
Piecewise Linear Functions (2)

- Space of continuous piecewise linear functions
  \[ V_n = \text{span} (\Phi_n) \]
- Interpolants \( u_n \in V_n \)
  \[
  u_n(x) = \sum_{i=0}^{2^n} \alpha_{n,i} \phi_{n,i}(x)
  \]
- \( V_n \) the space of all such interpolants \( u_n \)
Composite Simpson’s Rule: Function

**Interpolant**

- Continuous, piecewise quadratic function
- More complicated basis:

- Basis functions: Lagrangian polynomials, glued together
- $\alpha_i$: function values at grid points
- Basis functions have area $h/6$ (blue), $4h/6$ (red), $2h/6$ (green)
- We’ll not formally define basis functions here...
From Composite Trapezoidal to Archimedes

Piecewise linear functions

- We restrict our functions $u$ to $u(0) = u(1) = 0$
- Nodal point basis for discretization level $n$:
  \[
  \Phi_n := \{ \phi_{n,i}, 1 \leq i \leq 2^n - 1 \}
  \]
- Wanted: function space
  \[
  \mathcal{V} := \bigcup_{l=1}^{\infty} \mathcal{V}_l
  \]
  contains all functions which are in $\mathcal{V}_l$ for sufficiently large $l$
- However: generating system of $\mathcal{V}$ as
  \[
  \Phi := \bigcup_{l=1}^{\infty} \Phi_l
  \]
  does not lead to a basis (not linear independent)
Hierarchical Basis

- We are interested in a hierarchical decomposition of $V_l$
  - Define **hierarchical increment** $W_l$, such that $V_l$ is a *direct sum*:
    \[ V_l = V_{l-1} \oplus W_l \]

**Side-note: direct sum**
- Every $u_l \in V_l$ can be uniquely decomposed as
  \[ u_l = u_{l-1} + w_l, \text{ with } u_{l-1} \in V_{l-1} \text{ and } w_l \in W_l \]

- $W_l$ has to contain $2^{l-1}$ ansatz functions:
  \[ \dim V_l = 2^l - 1 = \dim V_{l-1} + \dim W_l \]
- This holds (introducing index sets $\mathcal{I}_l$) for
  \[ \mathcal{I}_l := \{ i : 1 \leq i < 2^l, \; i \text{ odd} \} \]
  \[ W_l := \text{span} \{ \phi_{l,i} : i \in \mathcal{I}_l \} \]
Hierarchical Increments

- Set of hierarchical increments $W_l$
- For $l = 1$: $W_1 = V_1$
- Example for $l = 1, 2, 3$: 

\[
\begin{align*}
\Phi_{1,1} \\
\Phi_{2,1} & \quad \Phi_{2,3} \\
\Phi_{3,1} & \quad \Phi_{3,3} & \quad \Phi_{3,5} & \quad \Phi_{3,7} \\
x_{1,1} & \quad x_{2,1} & \quad x_{2,3} & \quad x_{3,1} & \quad x_{3,3} & \quad x_{3,5} & \quad x_{3,7}
\end{align*}
\]
Hierarchical Basis (cont.)

- Then

\[ V_n = \bigoplus_{l=1}^{n} W_l \]

is a direct sum, too:

- \( u \in V_n \) can be decomposed uniquely into \( w_l \in W_l \):

\[ u = \sum_{l=1}^{n} w_l = \sum_{l=1}^{n} \sum_{i \in \mathcal{I}_l} v_{l,i} \phi_{l,i} \]

\( \rightarrow \) Coefficients \( v_{l,i} \) are hierarchical surplusses

- Corresponding basis of \( V_n \) (or, with \( \infty \) instead of \( n \), of \( V \))

\[ \Psi_n := \bigcup_{l=1}^{n} \{ \phi_{l,i} : i \in \mathcal{I}_l \} \]
Comparison

\[ l = 1 \]

\[ l = 2 \]

\[ l = 3 \]
Comparison (2)

\[ f(x) \]

\[ u(x) = \sum_{i} \alpha_i \phi_i(x) \]

\[ h_3 = 2^{-3} \]

\[ x_i \]

\[ 0 \]

\[ 1 \]
Numerical Integration with Hierarchical Basis
– Key Ingredients –

- Integration of $u(x)$:

$$
\int_a^b u(x) \, dx = \sum_{i=1}^{\frac{n}{2}} \alpha_i \int_a^b \phi_i(x) \, dx,
$$

- Using a hierarchical basis:

$$
\int_a^b u \, dx = \int_a^b \sum_{l=1}^{n} \sum_{i \in \mathcal{I}_l} v_{l,i} \phi_{l,i} \, dx = \sum_{l=1}^{n} \sum_{i \in \mathcal{I}_l} v_{l,i} \int_a^b \phi_{l,i} \, dx = \sum_{l=1}^{n} \sum_{i \in \mathcal{I}_l} v_{l,i} h_l
$$

- Computation of hierarchical surpluses:

$$
v_{l,i} = u(x_{l,i}) - \frac{1}{2} \left( u(x_{l,i-1}) + u(x_{l,i+1}) \right)
$$

i.e., difference between function and linear interpolant (on coarser level) at $x_{l,i} \rightarrow$ hierarchical surplus