

Algorithms of Scientific Computing (Algorithmen des Wissenschaftlichen Rechnens)

Haar Wavelets and Wavelet Transform

The wavelet families we look at (e.g. Haar wavelets) are constructed around a *multiresolution analysis*, a nested sequence V_n of function spaces some of which properties are

$$V_j \subset V_{j+1}, j \in \mathbb{Z} \quad (1)$$

$$\bigcap_{j=-\infty}^{\infty} V_j = \{0\} \quad (2)$$

$$f(t) \in V_l \Leftrightarrow f(2^{-l}t) \in V_0 \quad (3)$$

$$\begin{aligned} V_l &= V_{l-1} \oplus W_{l-1} \\ &= V_{l-2} \oplus W_{l-2} \oplus W_{l-1} \\ &= V_0 \oplus W_0 \oplus W_1 \oplus \dots \oplus W_{l-1}, \end{aligned} \quad (4)$$

with *orthogonal* functions $f \in V_j$ and $g \in W_j$, i.e. $\langle f, g \rangle = 0$.

The theory of multiresolution analysis further states the existence of a unique function ϕ which satisfies a so-called *dilation equation* of the form

$$\phi(t) = \sum_{k \in \mathbb{Z}} c_k \cdot \phi(2t - k). \quad (5)$$

Define another function, known as the **mother wavelet** or the **wavelet function** of the form

$$\psi(t) := \sum_{k \in \mathbb{Z}} (-1)^k c_{1-k} \cdot \phi(2t - k). \quad (6)$$

With the help of ϕ and ψ , we can define *orthonormal nodal bases* $\{ \phi_{l,k} \}$ for V_l with

$$\begin{aligned} \phi_{l,k}(t) &= \phi(2^l t - k) \\ \text{span}\{ \phi_{l,k} \} &= V_l, \quad \langle \phi_{l,k}, \phi_{l,m} \rangle = \delta_{k,m} \quad k, m \in \mathbb{Z}. \end{aligned} \quad (7)$$

The function ϕ is called **father wavelet** or the **scaling function**, and together with a **mother wavelet** ψ , they define the wavelet family. It is not necessary to know a specific formula for ϕ , the dilation equation (5) with its coefficients c_k together with the theory of multiresolution analysis provide enough information to derive the mother wavelet ψ as well as *orthonormal wavelet bases* $\{ \psi_{l,m} \}$ for the W_l with

$$\begin{aligned} \psi_{l,k}(t) &= \psi(2^l t - k) \\ \text{span}\{ \psi_{l,k} \} &= W_l, \quad \langle \psi_{l,k}, \psi_{l,m} \rangle = \delta_{k,m} \quad k, m \in \mathbb{Z}. \end{aligned} \quad (8)$$

Exercise 1: Cranking the Machine

Typically the scaling function ϕ is not known explicitly, and sometimes a closed-form analytic formula does not even exist. However, for continuous ϕ we can approximate the function to arbitrarily high precision using the “Cascade Algorithm”, a fixed-point method for functions.

In this exercise we want to implement this algorithm by iterating over the expression

$$F(\gamma)(t) = \sum_k c_k \cdot \gamma(2t - k) \quad (9)$$

in order to find the fixed point γ of F . That is, at iteration n

$$\gamma_{n+1}(t) = \sum_k c_k \cdot \gamma_n(2t - k) \quad (10)$$

Our starting point γ_0 will be the hat function

$$\gamma_0(t) = \max\{1 - |x|, 0\}. \quad (11)$$

- (i) Over the interval $[-1, 4]$ plot the approximations of the scaling function ϕ for the Haar wavelet family obtained in the first 7 iterations of the cascade algorithm. Do so by plugging the refinements coefficients c_k , $k = 0, 1$ in (12) into (9) resp. (5).

$$c_0 = c_1 = 1 \quad (12)$$

- (ii) Over the interval $[-1, 4]$ plot the approximations of the scaling function ϕ for the Daubechies wavelet family obtained in the first 7 iterations of the cascade algorithm. Do so by plugging the refinements coefficients c_k , $k = 0, \dots, 3$ in (13) into (9) resp. (5).

$$c_0 = \frac{1 + \sqrt{3}}{4} \quad c_1 = \frac{3 + \sqrt{3}}{4} \quad c_2 = \frac{3 - \sqrt{3}}{4} \quad c_3 = \frac{1 - \sqrt{3}}{4} \quad (13)$$

Exercise 2: The Haar Wavelet Basis

We derive the *mother wavelet* ψ as well as *orthonormal wavelet bases* $\{\psi_{l,m}\}$ with

$$\begin{aligned} \psi_{l,k}(t) &= \psi(2^l t - k) \\ \text{span}\{\psi_{l,k}\} &= W_l, \quad \langle \psi_{l,k}, \psi_{l,m} \rangle = \delta_{k,m} \quad k, m \in \mathbb{Z}. \end{aligned} \quad (14)$$

In this exercise we want to compute the 1- d wavelet transform for the Haar wavelet family and apply it to a signal vector \vec{s} of length $m = 2^n$. The transform can be implemented very efficiently as a “pyramidal algorithm” taking $\mathcal{O}(m)$ steps. For educational purpose we focus on the $\mathcal{O}(m^2)$ matrix-based algorithm.

- (i) Write a function that constructs the transformation matrix M consisting of the basis vectors $\psi_{l,k}$, $l \leq n$, $0 \leq k \leq 2^n - 1$.
- (ii) Use Python’s package `numpy.linalg` to invert the matrix.
- (iii) Use the program to compute the transform $\vec{d} = M^{-1}\vec{s}$ as well as the reconstructed signal $\vec{s} = M\vec{d}$ of the vector

$$\vec{s} = [1, 2, 3, -1, 1, -4, -2, 4]^T$$

- (iv) Verify the program’s output tracing the steps by hand.

Exercise 3: Discrete Wavelet Transform

Compute the DWT for the Haar wavelets for the signal $s = [8, 4, -1, 1, 0, 4, 1, 7, -\frac{5}{2}, -\frac{3}{2}, 0, -4, -2, -2, 1, -5]$ using the Pyramidal Algorithm. Discuss the computation complexity of this method.

Exercise 4: Discrete Wavelet Transform 2D

Compute the DWT for the Haar wavelets for the 2D signal $s = \begin{bmatrix} 4 & 2 & 3 & 5 \\ 1 & -7 & 0 & 8 \\ -1 & -3 & 9 & -3 \\ 6 & -2 & -1 & 1 \end{bmatrix}$.