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WS 2002/2003
Worksheet 4

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Fundamental Algorithms

Exercise 1

Consider the following RAM program:

```
0    R0 ← 1
1    R1 ← 8
2    R2 ← 1
3    R8 ← R8 + R0
4    R8 ← R8 – R0
5    R3 ← R8 – R2
6    IF R3 = 0 GOTO 24
7    R4 ← R2
8    R5 ← R2 + R0
9    R3 ← R5 – R8
10   IF R3 > 0 GOTO 4
11   R4 ← R4 + R1
12   R5 ← R5 + R1
13   R6 ← RR4
14   R7 ← RR5
15   R3 ← R6 – R7
16   IF R3 = 0 GOTO 19
17   RR4 ← R7
18   RR5 ← R6
19   R4 ← R4 – R1
20   R5 ← R5 – R1
21   R4 ← R4 + R0
22   R5 ← R5 + R0
23   GOTO 9
```

For an input sequence $(x_1, x_2, \ldots, x_n)$, the starting configuration of the RAM shall be given by:

$$\begin{aligned} \langle \text{R}i \rangle &:= 0 && \text{for } 0 \le i \le 7 \quad \text{and} \quad i \ge n+9 \\ \langle \text{R8} \rangle &:= n \\ \langle \text{R}i \rangle &:= x_{i-8} && \text{for } 9 \le i \le n+8 \\ \langle \text{PC} \rangle &:= 0 \end{aligned}$$

After the RAM has finished, registers R9 to R$(n+8)$ contain the input sequence in sorted (ascending) order.

Exercises:

(a) Execute the RAM program for the input sequence $(3, 7, 3, 2)$, and denote the contents of the registers for each step (steps where different registers are modified one after another may be combined into one step).

(b) What is the task of register R5?

(c) The given RAM program is a variant of one of the sorting algorithms discussed in the lectures. State the name of this algorithm, and discuss how this version differs from the version given in the lectures.

Solution

(a) The work steps performed by the RAM for the given input sequence are shown in table 1.

(b) R4 and R5 are loop variables that contain the respective array indices of the elements that are to be compared. If the elements have to be accessed, R4 and R5 are increased by 8, such that they contain the respective numbers of the registers where the array elements are stored. As $\langle R5 \rangle = \langle R4 \rangle + 1$, R5 always points to the right neighbour of the element pointed to by R4.

(c) The RAM program is a variant of Bubblesort, where the large numbers are transported to the end of the array (whereas in the variant discussed in the lectures, the small numbers where transported to the beginning of the array).

statement	R0	R1	R2	R3	R4	R5	R6	R7	R8	R9	R10	R11	R12
start	0	0	0	0	0	0	0	0	4	3	7	3	2
0 – 3	1	8	1	0	0	0	0	0	5	3	7	3	2
4 – 6	1	8	1	3	0	0	0	0	4	3	7	3	2
7 – 10	1	8	1	0	1	2	0	0	4	3	7	3	2
11 – 16	1	8	1	0	9	10	3	7	4	3	7	3	2
19 – 20	1	8	1	0	1	2	3	7	4	3	7	3	2
21 – 23	1	8	1	0	2	3	3	7	4	3	7	3	2
9 – 10	1	8	1	0	2	3	3	7	4	3	7	3	2
11 – 16	1	8	1	4	10	11	7	3	4	3	7	3	2
17 – 18	1	8	1	4	10	11	7	3	4	3	3	7	2
19 – 20	1	8	1	4	2	3	7	3	4	3	3	7	2
21 – 23	1	8	1	4	3	4	7	3	4	3	3	7	2
9 – 10	1	8	1	0	3	4	7	3	4	3	3	7	2
11 – 16	1	8	1	5	11	12	7	2	4	3	3	7	2
17 – 18	1	8	1	5	11	12	7	2	4	3	3	2	7
19 – 20	1	8	1	5	3	4	7	2	4	3	3	2	7
21 – 23	1	8	1	5	4	5	7	2	4	3	3	2	7
9 – 10	1	8	1	1	4	5	7	2	4	3	3	2	7
4 – 6	1	8	1	2	4	5	7	2	3	3	3	2	7
7 – 10	1	8	1	0	1	2	7	2	3	3	3	2	7
11 – 16	1	8	1	0	9	10	3	3	3	3	3	2	7
19 – 20	1	8	1	0	1	2	3	3	3	3	3	2	7
21 – 23	1	8	1	0	2	3	3	3	3	3	3	2	7
9 – 10	1	8	1	0	2	3	3	3	3	3	3	2	7
11 – 16	1	8	1	1	10	11	3	2	3	3	3	2	7
17 – 18	1	8	1	1	10	11	3	2	3	3	2	3	7
19 – 20	1	8	1	1	2	3	3	2	3	3	2	3	7
21 – 23	1	8	1	1	3	4	3	2	3	3	2	3	7
9 – 10	1	8	1	1	3	4	3	2	3	3	2	3	7
4 – 6	1	8	1	1	3	4	3	2	2	3	2	3	7
7 – 10	1	8	1	0	1	2	3	2	2	3	2	3	7
11 – 16	1	8	1	1	9	10	3	2	2	3	2	3	7
17 – 18	1	8	1	1	9	10	3	2	2	2	3	3	7
19 – 20	1	8	1	1	1	2	3	2	2	2	3	3	7
21 – 23	1	8	1	1	2	3	3	2	2	2	3	3	7
9 – 10	1	8	1	1	2	3	3	2	2	2	3	3	7
4 – 6	1	8	1	0	2	3	3	2	1	2	3	3	7
24 (STOP)	1	8	1	0	2	3	3	2	1	2	3	3	7

Table 1: work steps of the given RAM for the input sequence $(3, 7, 3, 2)$

Exercise 2

Show that $V_{\text{Merge}}^{\text{rek}}(n) = n \cdot \lceil \log n \rceil - 2^{\lceil \log n \rceil} + 1$ will solve the MERGESORT recurrence

$$\begin{aligned} V_{\text{Merge}}^{\text{rek}}(1) &= 0, \\ V_{\text{Merge}}^{\text{rek}}(n) &= V_{\text{Merge}}^{\text{rek}}\left(\left\lceil \frac{n}{2} \right\rceil\right) + V_{\text{Merge}}^{\text{rek}}\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n - 1 \quad \text{for } n > 1 \end{aligned}$$

Solution:

For $n = 1$, we get:

$$1 \cdot \lceil \log 1 \rceil - 2^{\lceil \log 1 \rceil} + 1 = 1 \cdot 0 - 2^0 + 1 = 0$$

For $n > 1$, we substitute $V_{\text{Merge}}^{\text{rek}}(n) = n \cdot \lceil \log n \rceil - 2^{\lceil \log n \rceil} + 1$ in the recurrence, and check whether it is satisfied:

case $n = 2^k$**:** left hand side:

$$V_{\text{Merge}}^{\text{rek}}(2^k) = 2^k \cdot k - 2^k + 1 = 2^k(k-1) + 1,$$

and for the right hand side, we get:

$$\begin{aligned} V_{\text{Merge}}^{\text{rek}}(2^{k-1}) + V_{\text{Merge}}^{\text{rek}}(2^{k-1}) + 2^k - 1 &= 2 \cdot \left(2^{k-1}(k-1) - 2^{k-1} + 1\right) + 2^k - 1 \\ &= 2^k(k-1) + 1 \end{aligned}$$

case $n = 2m$**, and** $2^k < 2m < 2^{k+1}$**:** then, $\lceil \log m \rceil = k$, and $\lceil \log 2m \rceil = k+1$, and therefore:

$$V_{\text{Merge}}^{\text{rek}}(2m) = 2m \cdot (k+1) - 2^{k+1} + 1 = 2^k(k-1) + 1,$$

while the right hand side computes to

$$\begin{aligned} V_{\text{Merge}}^{\text{rek}}(m) + V_{\text{Merge}}^{\text{rek}}(m) + 2m - 1 &= 2 \cdot \left(m \cdot k - 2^k + 1\right) + 2m - 1 \\ &= 2mk - 2^{k+1} + 2 + 2m - 1 \\ &= 2m \cdot (k+1) - 2^{k+1} + 1 \end{aligned}$$

case $n = 2m+1$**, and** $2^k < 2m+1 < 2^{k+1}$**:** then, $\left\lceil \log\left(\frac{2m+1}{2}\right) \right\rceil = k$, and $\lceil \log(2m+1) \rceil = k+1$. Consequently:

$$V_{\text{Merge}}^{\text{rek}}(2m+1) = (2m+1) \cdot (k+1) - 2^{k+1} + 1 = 2^k(k-1) + 1.$$

For the right hand side, we get

$$\begin{aligned} V_{\text{Merge}}^{\text{rek}}(m+1) + V_{\text{Merge}}^{\text{rek}}(m) + (2m+1) - 1 &= (m+1)\lceil \log(m+1) \rceil - 2^{\lceil \log(m+1) \rceil} + 1 \\ &+ \ m + \lceil \log m \rceil - 2^{\lceil \log m \rceil} + 1 + 2m \end{aligned}$$

Here, we have to discriminate the two cases

1. $m > 2^{k-1}$, then $\lceil \log m \rceil = k$, and:

$$\begin{aligned} \ldots &= (m+1)\cdot k - 2^k + 1 + m \cdot k - 2^k + 1 + 2m \\ &= 2mk + k + 2m - 2^{k+1} + 2 \\ &= (2m+1)(k+1) - 2^{k+1} + 1 \end{aligned}$$

2. $m = 2^{k-1}$, then $\lceil \log m \rceil = k - 1$, and:

$$\begin{aligned} \ldots &= (m+1)\cdot k - 2^k + 1 + m \cdot (k-1) - 2^{k-1} + 1 + 2 \cdot 2^{k-1} \\ &= mk + k - 2m + 1 + mk - m - m + 1 + 2m = 2mk + k - 2m + 2 \\ &= 2mk + k + 2m - 4m + 2 = 2mk + k + 2m - 4 \cdot 2^{k-1} + 2 \\ &= 2mk + k + 2m - 2^{k+1} + 2 = (2m+1)(k+1) - 2^{k+1} + 1 \end{aligned}$$