Fundamental Algorithms 1

– Solution Examples –

Exercise 1

Prove (by induction over \( n \)) that \( \frac{1}{3}n^2 + 5n + 30 \in O(n^2) \) for all \( n \in \mathbb{N}^+ \).

Solution:
(Note: we wouldn’t have to prove by induction, but it’s a simple case to practice it.)

\[
f := \frac{1}{3}n^2 + 5n + 30 \in O(n^2) \iff \exists c > 0 \exists n_0 \forall n \geq n_0 : f(n) \leq cn^2
\]

Let \( c := 100, n_0 := 1 \).

Base case: \( n = n_0 = 1 \)

\[
\frac{1}{3} + 5 + 30 = \frac{35}{3} \leq 100
\]

Induction hypothesis: For some \( n \in \mathbb{N} \): \( f(n) \leq 100n^2 \)

Inductive step:

\[
f(n + 1) = \frac{1}{3}(n + 1)^2 + 5(n + 1) + 30
\]
\[
= \frac{1}{3}(n^2 + 2n + 1) + 5(n + 1) + 30
\]
\[
= f(n) + \frac{2}{3}n + \frac{16}{3}
\]
\[
\leq 100n^2 + \frac{2}{3}n + \frac{16}{3}
\]
\[
\leq 100n^2 + 200n + 100
\]
\[
= 100(n + 1)^2
\]

q.e.d.

Note: we chose a pretty large \( c \) for this prove – you should re-do this proof with smaller values for \( c \) (such as \( c = 1 \)) and see what happens.
Exercise 2

(a) Compare the growth of the following functions using the $o$, $O$, and $\Theta$-notation:

1. $n \log n$
2. $n^l$ for all $l \in \mathbb{N}$
3. $2^n$

(b) Try to give a simple characterization of the growth of the following expressions using the $\Theta$-notation:

1) $\sum_{i=1}^{n} \frac{1}{i}$
2) $\log(n!)$

Hint for $\log(n!)$: try to prove $n^2 \leq n! \leq n^n$ first!

Solution:

(a) $n^l \in o(2^n)$ for all $l \in \mathbb{N}$, because by de l'Hôpital's rule:

$$\lim_{n \to \infty} \frac{n^l}{2^n} = \lim_{n \to \infty} \frac{l \cdot n^{l-1}}{2^n \cdot \ln 2} = \lim_{n \to \infty} \frac{l \cdot (l-1) \cdot n^{l-2}}{2^n \cdot (\ln 2)^2} = \ldots = \lim_{n \to \infty} \frac{l!}{2^n \cdot (\ln 2)^l} = 0$$

Therefore, $n^l \in O(2^n)$ for all $l \in \mathbb{N}$.

Obviously, $n^1 \in o(n \log n)$ and $n^1 \in \Theta(n \log n)$, but for $l \geq 2$:

$$\lim_{n \to \infty} \frac{n \ln n}{n^l} = \lim_{n \to \infty} \frac{\ln n}{n^{l-1}} = \lim_{n \to \infty} \frac{1}{n \cdot (l-1) \cdot n^{l-2}} = 0$$

Therefore $n^l \in \omega(n \log n)$ for all $l \geq 2$. This also holds for any real $l > 1$.

As a consequence, $n \log n \in o(2^n)$.

(b) $\sum_{i=1}^{n} \frac{1}{i} \in \Theta(\log n)$:

Consider the functions $u(x) := \frac{1}{|x|}$ and $l(x) := \frac{1}{|x|}$, then:

$$l(x) \leq \frac{1}{x} \leq u(x) \Rightarrow \int_{1}^{n} l(x) \, dx \leq \int_{1}^{n} \frac{1}{x} \, dx \leq \int_{1}^{n} u(x) \, dx$$

$$\Rightarrow \sum_{i=2}^{n} \frac{1}{i} \leq \ln n - \ln 1 \leq \sum_{i=1}^{n} \frac{1}{i}$$

(draw a graph of $u(x)$ and $l(x)$ to see why the integrals are given by these sums).

Thus, $\ln n \leq \sum_{i=1}^{n} \frac{1}{i} \leq \sum_{i=1}^{n} \frac{1}{i}$, and therefore $\ln n \in O\left(\sum_{i=1}^{n} \frac{1}{i}\right)$.

As $2 \cdot \sum_{i=1}^{n} \frac{1}{i} = 2 \cdot \left(\frac{1}{2} + \cdots + \frac{1}{n}\right) > 1$, we know that

$$3 \sum_{i=2}^{n} \frac{1}{i} = 2 \sum_{i=2}^{n} \frac{1}{i} + \sum_{i=2}^{n} \frac{1}{i} > 1 + \sum_{i=2}^{n} \frac{1}{i} = \sum_{i=1}^{n} \frac{1}{i}$$
and, therefore,
\[
\sum_{i=1}^{n} \frac{1}{i} < 3 \sum_{i=2}^{n} \frac{1}{i} \leq 3 \ln n \quad \Rightarrow \quad \sum_{i=1}^{n} \frac{1}{i} \in O(\ln n), \quad \text{q.e.d.}
\]

2) Using \( n^2 \leq n! \leq n^n \), we get:
\[
\ln n^2 \leq \ln(n!) \leq \ln n^n \quad \Rightarrow \quad \frac{n}{2} \ln n \leq \ln(n!) \leq n \ln n,
\]
which leads directly to the result \( \ln(n!) \in \Theta(n \ln n) \).

**Proof** for \( n^2 \leq n! \leq n^n \):

It is obvious that \( n! = 1 \cdot 2 \cdot \ldots \leq n \cdot n \cdot \ldots = n^n \).

To prove \( n^2 \leq n! \), or \( n^n \leq (n!)^2 \), we show that \( \frac{(n!)^2}{n^n} \geq 1 \):

\[
\frac{(n!)^2}{n^n} = \frac{n!}{n^n} \cdot n! = \prod_{i=0}^{n-1} \frac{n-i}{n} \cdot \prod_{i=0}^{n-1} (i+1) = \prod_{i=0}^{n-1} \frac{(n-i)(i+1)}{n}
\]

and \((n-i)(i+1) = -i^2 + ni - i + n = n + i(n - 1 - i) \geq n\). Therefore, all factors of the product are \( \geq 1 \). Consequently, the product itself is \( \geq 1 \).

**Exercise 3**

Let \( l(x) \) be the number of bits of the representation of \( x \) in the binary system. Prove:
\[
\sum_{i=1}^{n} l(i) \in \Theta(n \log n)
\]

**Solution:**

We need the following equalities:

- \( \sum_{i=1}^{n} \log i = \log \left( \prod_{i=1}^{n} i \right) = \log(n!) \in \Theta(n \log n) \), (see exercise 1(b), part 2!), and

- \( l(i) = \lfloor \log_2 i \rfloor + 1 \) (if the binary representation of a number has \( l \) bits, the respective number \( i \) will be between \( 2^{l-1} \) and \( 2^l - 1 \)).

If we can show that
\[
c_1 \log_2 i \leq \lfloor \log_2 i \rfloor \leq \log_2 i
\]
for some constant \( 0 < c_1 < 1 \) (the second inequality is a trivial result of the definition of \( \lfloor \rfloor \)), and use the transformation
\[
\sum_{i=1}^{n} l(i) = \sum_{i=1}^{n} (\lfloor \log_2 i \rfloor + 1) = n + \sum_{i=1}^{n} \lfloor \log_2 i \rfloor,
\]
we get
\[
c_1 \left( n + \sum_{i=1}^{n} \log_2 i \right) \leq \sum_{i=1}^{n} l(i) \leq n + \sum_{i=1}^{n} \log_2 i \quad \Rightarrow \quad \sum_{i=1}^{n} l(i) \in \Theta(n \log n)
\]

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We still have to prove that $c_1 \log_2 i \leq \lfloor \log_2 i \rfloor$ for some $c_1$:
For $i \geq 3$, we can choose $c_1$, such that $i^{c_1} < \frac{i}{2}$ (choose $c_1 := \frac{1}{2}$, e.g.). Then
\[ c_1 \log_2 i = \log_2 (i^{c_1}) < \log_2 \frac{i}{2} = \log_2 i - 1 < \lfloor \log_2 i \rfloor. \]

As the inequality is also correct for $i \in \{1, 2\}$, we are finished.

**Exercise 4**

Prove that $\Theta$ defines an equivalence relation on the set of functions $\{f \mid f: \mathbb{N} \rightarrow \mathbb{R}\}$. Use that $(f, g) \in \Theta \iff f \in \Theta(g)$

**Solution:**

We define the relation $\Theta$ by $(f, g) \in \Theta :\iff f \in \Theta(g)$.

To show that $\Theta$ is an equivalence relation, we have to prove that:

- $\Theta$ is reflexive:
  - as $f \in \Theta(f)$ (e.g., choose constants $c_1 := \frac{1}{2}$, and $c_2 := \frac{3}{2}$), by definition $(f, f) \in \Theta$;

- $\Theta$ is symmetric:
  - if $f \in \Theta(g)$, then
    \begin{align*}
    f \in O(g) & \Rightarrow g \in \Omega(f) \\
    f \in \Omega(g) & \Rightarrow g \in O(f)
    \end{align*}
  
  Therefore, by definition $g \in \Theta(f)$;

- $\Theta$ is transitive:
  - if $f \in \Theta(g)$, and $g \in \Theta(h)$, then, there are constants $c_1$, $c_2$, $c_3$, and $c_4$, such that for sufficiently large $n$
    \begin{align*}
    c_1 f(n) & \leq g(n) \leq c_2 f(n) \\
    c_3 g(n) & \leq h(n) \leq c_4 g(n)
    \end{align*}
  
  Therefore, $c_1 c_3 f(n) \leq h(n) \leq c_2 c_4 h(n)$ which leads to $f \in \Theta(h)$. 