

1 Further notes on relaxed Jacobi

Stationary methods (such as Jacobi) reduce high-frequency components rapidly, but low-frequency components slowly. They are sometimes called smoothers. ω -JAC:

$$x^{(n+1)} = R_\omega x^{(n)} + \omega D^{-1}b.$$

Eigenvectors (EV) of A are EVs of iteration matrix R_ω . Let us consider

$$x^{(n+1)} = Rx^{(n)} + g \quad \text{here } g := \omega D^{-1}b. \quad (1)$$

If the iteration converges, it does not change the exact solution, i.e. it holds

$$x^* = Rx^* + g \quad (2)$$

with x^* being the exact solution. (1) - (2) yields

$$x^{(n+1)} - x^* = R(x^{(n)} - x^*)$$

which is

$$e^{(n+1)} = Re^{(n)}.$$

Therefore, the current error can be written as linear combination of EVs. Means that error consists of EVs of the iteration matrix R or R_ω for Jacobi:

$$e^{(n+1)} = R_\omega e^{(n)} = R_\omega \sum_{k=1}^n c_k \underbrace{q_k}_{EV} = \sum_{k=1}^n \lambda_k c_k q_k.$$

For Jacobi: Eigenvalues of R_ω are analytically known: $\lambda_k = 1 - 2\omega \sin^2(\frac{\pi k}{2n})$. Therefore, the error is influenced by eigenvalues of R_ω and the parameter ω . With "good" choice of ω many parts of the error are damped to zero.

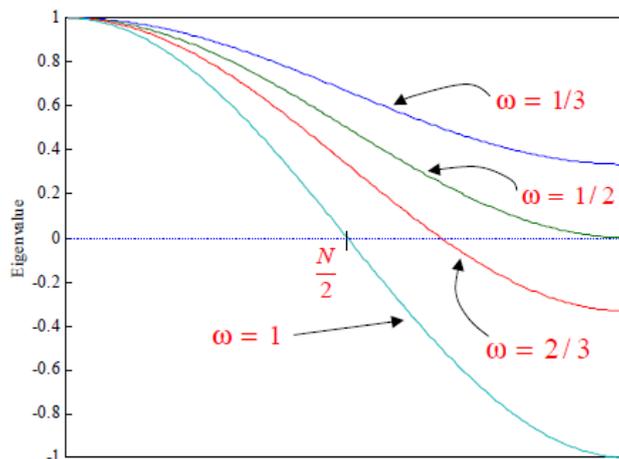


Figure: Eigenvalues of the iteration matrix R_ω in dependence of parameter ω

- $\omega = 1$: damping only at $\frac{n}{2}$ but for high frequencies no damping (and negation). Jacobi converges very slowly.

- $\omega = \frac{1}{2}$: damping of high frequencies very well but everything else badly.
- $\omega = \frac{2}{3}$: Used in praxis. Near intersection good damping and high frequencies good as well. Low frequencies not very good.

The search for optimal ω is minimization problem of area under damping curves and can be computed.

Remedy using Multigrid: High-frequency components can be damped efficiently instead of low-frequency components. The error will contain only low-frequency parts after several iterations. By increasing the mesh-size h , a coarser grid is used such that low-frequency components become high-frequency components. Again smoothing via relaxed Jacobi can be performed. Usually several Restrictions used (V-cycle, W-cycle, etc. - see literature for Multigrid).