Introduction to Scientific Computing


1.) One way to examine the stability properties of a numerical (one-step) scheme for the computation of an approximate solution of ordinary differential equations is to consider the behaviour of the numerical solution when the scheme is applied to the initial value problem

\[ y' = \lambda \cdot y, \quad y(x = 0) = y_0 \]

where \( \lambda \leq 0 \). The method is said to be stable in this context if all the approximate solutions satisfy

\[ |y_k| \leq |y_0|. \]

a.) For positive step size \( h \) show that the maximum step size is bounded by

\[ h < |2/\lambda| \]

in the case of the explicit Euler method

\[ y_{k+1} = y_k + h \cdot f(x_k, y_k). \]

b.) Examine the stability of the implicit Euler method

\[ y_{k+1} = y_k + h \cdot f(x_{k+1}, y_{k+1}) \]

and the Adams-Moulton method

\[ y_{k+1} = y_k + 0.5 \cdot h \cdot [f(x_k, y_k) + f(x_{k+1}, y_{k+1})] \]

in the same way. Do you find stability restrictions and, if so, which ones?

2.) Consider the differential equation

\[ y'' + 20 \cdot y' + 19 \cdot y = 0 \]

together with the initial conditions

\[ y(x = 0) = 1 \quad \text{and} \quad \left. \frac{dy}{dx} \right|_{(x=0)} = -10. \]
a.) Find the exact solution of this initial value problem.

b.) Compute a numerical solution applying the explicit Euler method in the interval $0 \leq x \leq 2$. To do so substitute $\frac{du}{dx}$ by the new variable $z$ and rewrite the second order differential equation as two first order differential equations. Write a computer program in your preferred programming language. What is the maximal step size? Please hand in a single plot containing your approximate solutions for step sizes $h = 1/8$, $h = 1/12$, $h = 1/16$, and $h = 1/32$ as well as the exact solution. You do not need to attach your source code.