Scientific Computing

Eigenvalue Problems and Algebraic Models

Exercise 4: Economics (Model by Leontief)

In our city, we have \( n \) companies. Each company sells its own product \( x_j, j = 1, \ldots, n \). Besides, in order to produce one unit of its product, the company \( j \) requires \( A_{ij} \) units of the product from company \( i \). The matrix \( A_{ij}, i, j = 1, \ldots, n \) hence describes the production relations within our city.

(a) **Closed economic system**: Assume that our city is not involved in any trades with companies from other cities (no export or import). Each company hence uses all its incomes for further productions. How can we estimate the amount of products \( x_i \) which are required to have a stable economic system (that is everything that is produced is used for further productions again)?

(b) Consider the production matrix:

\[
A := \begin{pmatrix}
\frac{3}{4} & 1 & \frac{19}{8} \\
\frac{1}{2} & \frac{3}{4} & \frac{1}{4} \\
0 & 0 & 2
\end{pmatrix}
\] (1)

Compute its eigenvalues and eigenvectors. Can the respective city–based on this production matrix–potentially survive as a closed economic system?

(c) Since a new company will enter the city market in two years, the companies from (b) have to provide \( x = (1.5, 1, 0) \) products until then. Assume that the production matrix denotes the requirements that need to be fulfilled within one year. How many products do the companies need to provide at the present stage according to the Leontief model? How does this behaviour relate to the eigenvalues of our system?

(d) **Import and export**: Our city is now allowed to buy and sell products from/to other companies from outside. How can we modify our model to include this property? For which vector \( x \) do we then obtain a stable solution for the economic state of our city?
Exercise 5: Sport Rankings

Alice and Bob like watching baseball games. They watch approx. one match per day. Since they are very big fans, they know all about the statistics of the teams (i.e. how often did team $i$ win against $j$ and vice versa; there are no draws).

(a) Bob likes those teams who win. If his current favorite team wins, he will also watch this team’s next match on the following day. If the team loses, Bob changes his mind and watches the match of the respective winner team. How often will Bob stay with one team in future?

(b) With a probability $\beta$, a match needs to be canceled due to bad weather. Similar to Bob, Alice also watches the matches of the winner teams (hence, if her favorite team wins, she normally stays with this team as well and otherwise switches to the new winner team). If there is bad weather, Alice randomly chooses the team and will watch this team’s match the next day. How long will Alice stay with each team?