Scientific Computing

Ordinary Differential Equations: Numerical Methods

Exercise 13: Convergence of the Euler Method

Consider the ODE

\[
\frac{dy(t)}{dt} = Ay(t) + b
\]  

with \( A \in \mathbb{R}^{N \times N}, y(t) : \mathbb{R}^N \to \mathbb{R}^N \) and \( b \in \mathbb{R}^N \) (this could for example be the linear system arising from the two-species model). The explicit Euler method applied to this equation reads:

\[
y^{(n+1)} = y^{(n)} + \tau (Ay^{(n)} + b)
\]

with time step \( \tau \) and \( y^{(n)} := y(n \cdot \tau) \).

(a) Show the following statement: if the Euler method converges towards a vector \( y^* \), then \( y^* \) must be a critical point of the ODE.

(b) Under which conditions does the Euler discretisation from above (Eq. (2)) converge towards a critical point \( y^* \)?

Exercise 14: Analysis of Single-Step Methods

Consider the ODE from last time

\[
\frac{d^2 y}{dt^2} = -y
\]

and its transform into a first-order system of ODEs

\[
\begin{pmatrix}
\frac{dy_0(t)}{dt} \\
\frac{dy_1(t)}{dt}
\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot 
\begin{pmatrix} y_0(t) \\ y_1(t) \end{pmatrix}
\]  

(a) Formulate the discrete update rule for the first-order system of Eq. (3) when applying the following single-step methods and using a time step \( \tau \):

- explicit Euler method
- implicit Euler method
trapezoidal rule (Crank-Nicolson)

Write down the respective update scheme in matrix-vector form as

\[
\begin{pmatrix}
y_0^{n+1} \\
y_1^{n+1}
\end{pmatrix} = \mathbf{A}_{\text{method}} \cdot \begin{pmatrix}
y_0^n \\
y_1^n
\end{pmatrix}
\]

where \( \mathbf{A}_{\text{method}} \) denotes the method- and time step-dependent matrix for each of the single-step methods from above and \( y^n := y(n \cdot \tau) \). What can you say about the long-time behaviour of the system, that is for \((y_0^n, y_1^n)\) when \( n \to \infty \)?

(b) Write a maple sheet and check your analytical findings. You may consider solving the ODE from Eq. (3) for the initial values \( y(0) = 0, dy(0)/dt = 1 \).