Scientific Computing

Partial Differential Equations

Jacobi Method

An iterative method to solve linear systems of equations $Ax = b$ with $A \in \mathbb{R}^{N \times N}$, $b \in \mathbb{R}^N$ is given by the Jacobi method. Starting from an initial vector $x^{(0)}$, the iteration procedure reads:

$$
x^{(n+1)}_i = \frac{1}{A_{ii}} \left( b_i - \sum_{j \neq i} A_{ij} x^{(n)}_j \right), \quad i = 1, \ldots, N
$$

Convergence of the Jacobi Method: Diagonal dominance

A matrix $A \in \mathbb{R}^{N \times N}$ is *diagonally dominant* if the following inequality holds for all $i \in \{1, \ldots, N\}$:

$$
|A_{ii}| \geq \sum_{j \neq i} |A_{ij}|
$$

Diagonal dominance is an important property of matrices and helps to show that for example the Jacobi iteration scheme for $Ax = b$ converges:

- If $A$ is strictly diagonally dominant, that is we have a “$>$” in Eq. (1) for all rows $i$, then the Jacobi iteration converges.
- Let $A$ be diagonally dominant and have at least one row with “$>$” in Eq. (1). If $A$ is irreducible, then the Jacobi method converges.

We do not want to dive deeper into mathematics here and thus will not completely define what irreducibility means (we may discuss the respective definition during the exercise class). However, it should be noted that a matrix is irreducible if the matrix is tridiagonal and only has non-vanishing main- and subdiagonal entries, that is

$$
A_{ij} \neq 0 \quad \forall |i - j| \leq 1.
$$
Exercise 17: Convection-Diffusion Systems

Consider the one-dimensional differential equation

\[- \frac{d^2 u(x)}{dx^2} + v \frac{du(x)}{dx} = f(x), \quad x \in (0,1)\]

(2)

with velocity \( v \in \mathbb{R} \) and Dirichlet boundary conditions \( u(0) = c_0, u(1) = c_1 \). This equation models the transport of a quantity \( u \) in a fluid when the fluid is assumed to move at constant velocity \( v \).

(a) Set up a finite difference scheme using

- the standard second-order discretisation of the diffusive term (that is the second-order derivative)
- a symmetric second-order discretisation for the convective term (that is the first-order derivative).

Write down the formulation for a single row of the arising linear system of equations

\[ \sum_j A_{ij} u_j = b_i \]

with \( u_j := u(j \cdot h) \), meshsize \( h \) and right-hand side \( b_i \).

(b) Formulate the Jacobi relaxation to solve this system. Under which conditions does the Jacobi method converge?

(c) Replace the second-order discretisation of the first-order derivative by a first-order one-sided discretisation. Re-formulate the Jacobi relaxation for this case. Under which conditions can we expect convergence now?

Exercise 18: Runge-Kutta for the Heat Equation

The following partial differential equation describes the distribution of the temperature \( T \) in a stick:

\[ \frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2}, \quad x \in (0,1) \]

The constant \( D \) is the thermal diffusivity and describes how fast the temperature can diffuse within the stick. We further assume that the temperature of the stick at the outer ends is known, that is \( T(t, x = 0) = T_0, T(t, x = 1) = T_N \).

(a) Review from the lecture: Apply the symmetric finite difference approximation for the second-order spatial derivative similar to exercise 17(a). Which kind of differential equations remains?

(b) Discretise the new differential equations from (a) using the method of Heun.

(c) Write a maple sheet that solves the discrete problem with \( D = 1 \), a time step \( \tau = 0.001 \), an initial temperature distribution \( T(t = 0, x) = 0 \) in the inner part of the stick and temperature values \( T_0 = 0, T_1 = 1 \). Use a mesh size \( h = 1/20 \) for the spatial discretisation and plot the result after 1, 10, 100 and 1000 time steps.