2. Multigrid

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Multigrid - error frequency

Low-frequency errors on fine grids become high-frequency errors on coarse grids

- $k$-th eigen vector of iteration matrix $r$ generated by
  \[ v_j^k = \sin \left( \frac{kJ \pi}{n+1} \right) = \sin (k \pi \cdot j \cdot h) \]

- high frequent: $k > n/2$

- Example: wave with $k = 4$ on two grids

**Figure:** $v_j^4 = \sin \left( \frac{4 \cdot \pi \cdot j}{12} \right)$ on a grid with meshwidth $h = 1/12$.

**Figure:** $v_j^4 = \sin \left( \frac{4 \cdot \pi \cdot j}{6} \right)$ on a grid with meshwidth $h = 1/6$. 
Two-Grid - Correction Scheme

1. Relaxation step:
   \[ u_h^{(n)} = f_h \]
   
2. Compute fine-grid residual:
   \[ r_h = f_h - A_h v_h \]
   
3. Restrict residual to the coarse grid:
   \[ r_{2h} = R_{2h} h r_h \]
   
4. Solve/relax on the coarse grid:
   \[ A_{2h} e_{2h} = r_{2h} \]
   
5. Interpolate coarse grid error to the fine grid:
   \[ e_h = P_{2h} e_{2h} \]

6. Correct fine grid solution:
   \[ v_h = v_h + e_h \]

7. Relaxation step again:
   \[ u_h^{(n+1)} = f_h \]

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Scientific Computing II, Wolfgang Eckhardt
Two-Grid - Correction Scheme

- relax $n_1$ times on $A^h u^h = f^h$ on $\Omega^h$ with initial guess $v^h$

- compute fine-grid residual $r^h = f^h - A^h v^h$, restrict it to the coarse grid by $r^{2h} = R^h r^h$

- solve / relax on $A^{2h} e^{2h} = r^{2h}$ on the coarse grid

- interpolate coarse grid error to the fine grid by $e^h = P^h e^{2h}$

- correct fine grid solution by $v^h = v^h + e^h$

- relax $n_2$ times on $A^h u^h = f^h$ with corrected guess $v^h$. 
Two-Grid - Example

Standard example: 1d Poisson-Equation

\[-u'' = 0 \quad \text{in } ]0; 1[, \]
\[u = 0 \quad \text{at } \partial ]0; 1[.\]

\[h = \frac{1}{64} \quad \text{with initial guess}\]

\[x_j^h = \frac{1}{2} \left[ \sin \left( \frac{16j\pi}{n} \right) + \sin \left( \frac{40j\pi}{n} \right) \right]\]

We perform

- 3 Damped-Jacobi-iterations on the fine grid (presmoothing)
- 3 Damped-Jacobi-iterations on the coarse grid
- 3 Damped-Jacobi-iterations on the fine grid (postsmoothing)
Two-Grid - Example

After three smoothing steps, the high-frequency part of the error is smooth, the low-frequency part dominates.
Two-Grid - Example

**Figure:** After the coarse-grid correction, the low-frequency part is reduced drastically, mainly high-frequencies remain.

**Figure:** The high-frequency error is smoothed.
Multigrid - V-Cycle

Idea: Apply Correction-Scheme recursively, up to the coarsest grid level.

\( v^h = \text{procedure } \text{VCycle}(v^h, f^h) \)

1. Relax \( n_1 \) times on \( A^h u^h = f^h \) with given initial guess \( v^h \).
2. If \( \Omega^h = \text{coarsest grid} \) then goto step 4.
   Else
   \( f^{2h} = R^h \cdot \text{res}^h \),
   \( v^{2h} = 0 \),
   \( v^{2h} = \text{VCycle}(v^{2h}, f^{2h}) \).
3. Correct \( v^h = v^h + P_{2h}^h v^{2h} \)
4. Relax \( n_2 \) times on \( A^h u^h = f^h \) with corrected initial guess \( v^h \).