Divergence-Free Elements for Incompressible Flow on Cartesian Grids

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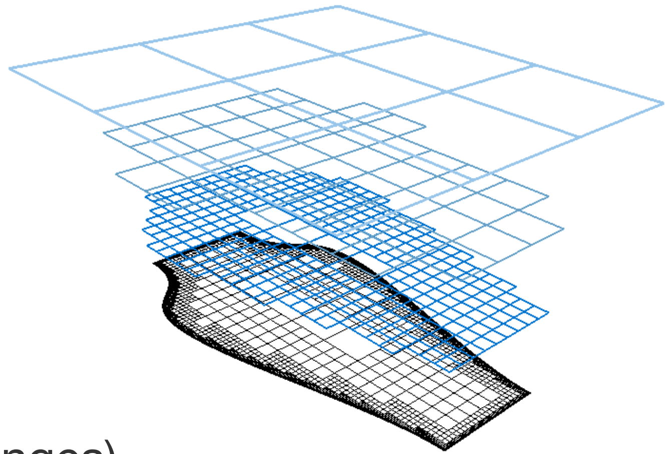
Outline

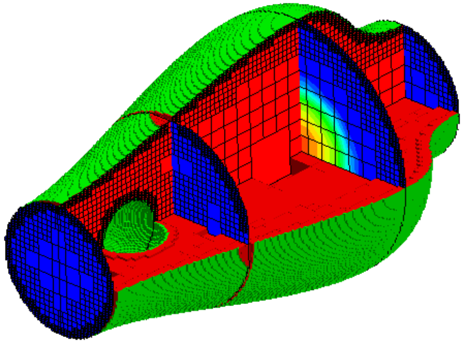
- The PDE Framework Peano
- Div-Free Elements
  - Motivation
  - Derivation
  - Validation
- Variants of Div-Free Elements
  - Enhancement
  - Rotated Coordinate System
- Summary & Outlook

The PDE Framework Peano

- Cartesian grids
  (recursive adaptivity, full grid hierarchy)
- Low memory requirements
- Space-filling curves + stack data structures
  ➔ high cache-hit rates (>98%)
  ➔ flexible insertion/deletion of data (grid changes)
- Shared/distributed mem. parallelisation
- Software engineering aspects



source: T. Weinzierl



- CFD component
  - Incompressible flow (FEM, IDO)
  - Explicit + implicit time-integration schemes (FE, RK4, BE, (adaptive) TR)

Motivation 1: Discrete Energy Conservation

$$\frac{d}{dt}\left(\mathbf{u}_h^T A \mathbf{u}_h\right) \quad + \quad \mathbf{u}_h^T \left(C(\mathbf{u}_h) + C(\mathbf{u}_h)^T\right) \mathbf{u}_h + \mathbf{u}_h^T \left(D + D^T\right) \mathbf{u}_h - \mathbf{u}_h^T M^T p_h - p_h^T M \mathbf{u}_h = 0$$

$$\Downarrow$$

$$\frac{d}{dt}\left(\mathbf{u}_h^T A \mathbf{u}_h\right) \quad = \quad -\mathbf{u}_h^T \left(C(\mathbf{u}_h) + C(\mathbf{u}_h)^T\right) \mathbf{u}_h - \mathbf{u}_h^T \left(D + D^T\right) \mathbf{u}_h$$

$$\text{no viscosity} \quad : \quad \frac{d}{dt}\left(\mathbf{u}_h^T A \mathbf{u}_h\right) \equiv 0 \qquad \Longleftarrow \qquad C(\mathbf{u}_h) \stackrel{!}{=} -C(\mathbf{u}_h)^T$$

$$\text{viscosity} \quad : \quad \frac{d}{dt}\left(\mathbf{u}_h^T A \mathbf{u}_h\right) \leq 0 \qquad \Longleftarrow \qquad D + D^T \text{pos. semi-definite!}$$

R. Verstappen & A. Veldman, J. Comput. Phys. 187, 2003

Motivation 1: Discrete Energy Conservation

$$\frac{d}{dt}\left(\mathbf{u}_h^T A \mathbf{u}_h\right) + \mathbf{u}_h^T\left(C(\mathbf{u}_h)+C(\mathbf{u}_h)^T\right)\mathbf{u}_h + \mathbf{u}_h^T\left(D+D^T\right)\mathbf{u}_h - \mathbf{u}_h^T M^T p_h - p_h^T M \mathbf{u}_h = 0$$

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$$\frac{d}{dt}\left(\mathbf{u}_h^T A \mathbf{u}_h\right) = -\mathbf{u}_h^T\left(C(\mathbf{u}_h)+C(\mathbf{u}_h)^T\right)\mathbf{u}_h - \mathbf{u}_h^T\left(D+D^T\right)\mathbf{u}_h$$

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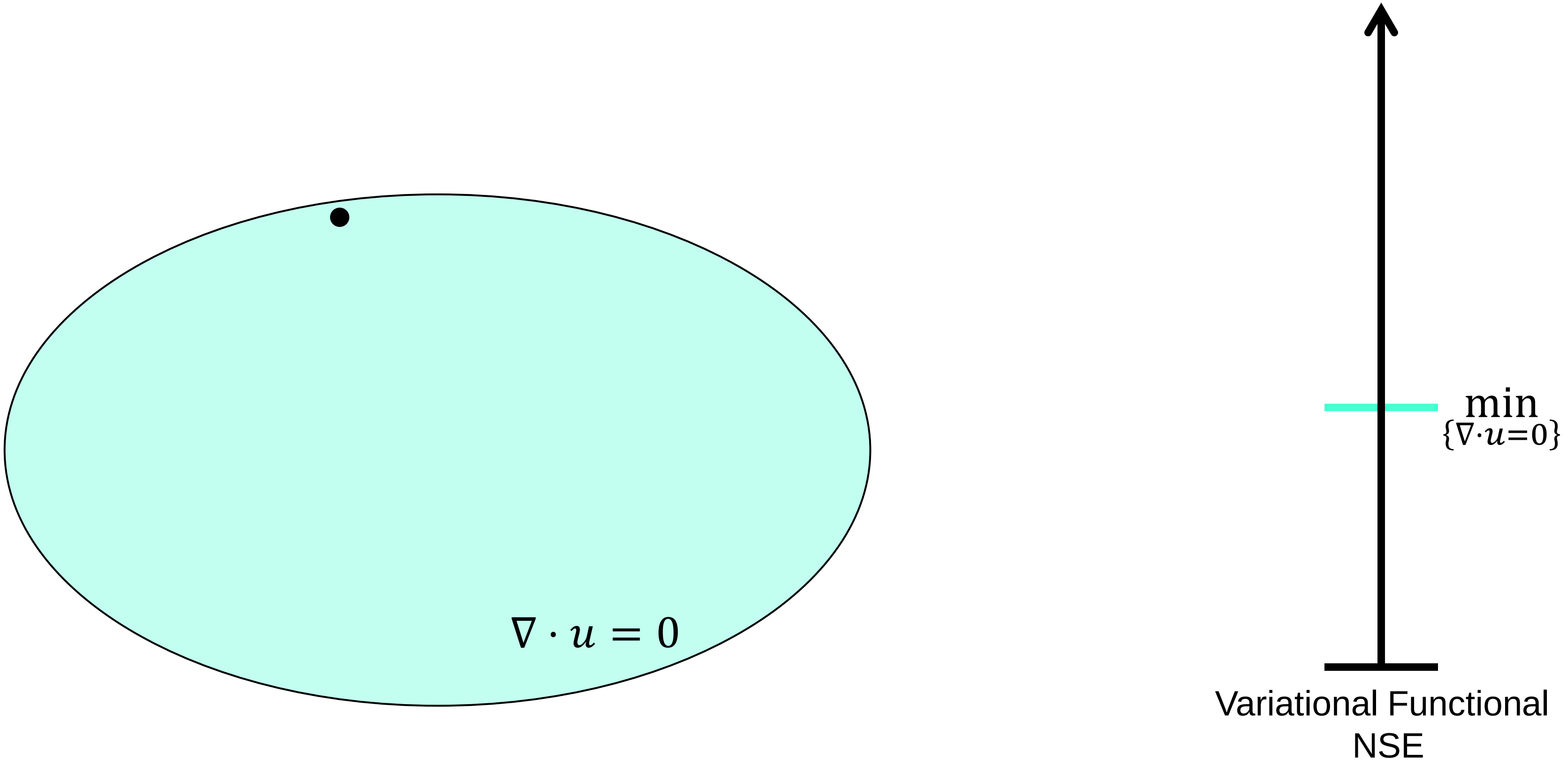
$$\text{viscosity} : \quad \frac{d}{dt}\left(\mathbf{u}_h^T A \mathbf{u}_h\right) \leq 0 \quad \Longleftarrow \quad D+D^T \text{pos. semi-definite!}$$

(skew-) symmetry ➔ guaranteed stable DAEs!

R. Verstappen & A. Veldman, J. Comput. Phys. 187, 2003

Motivation 2: FEM Function Spaces for NSE



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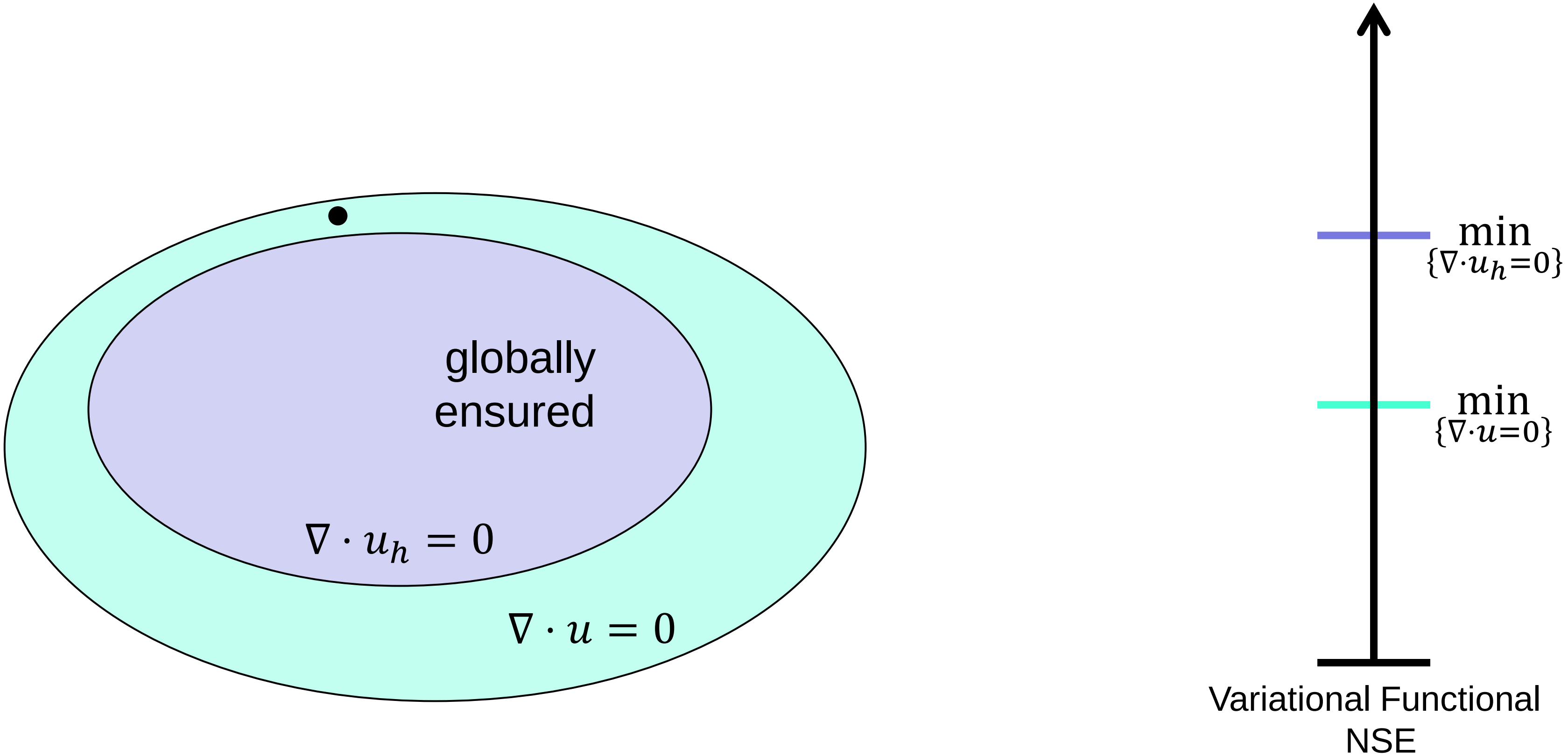

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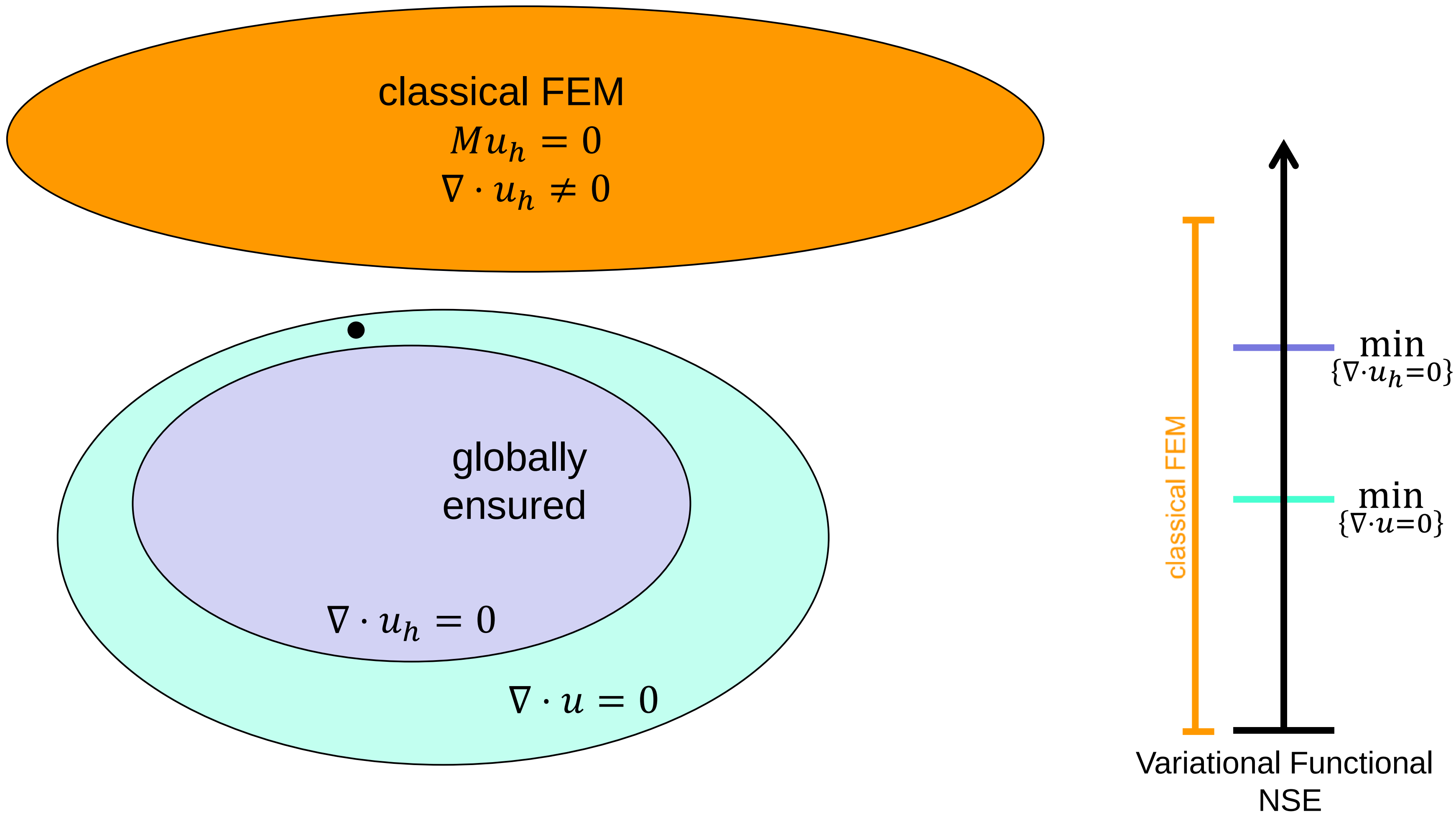

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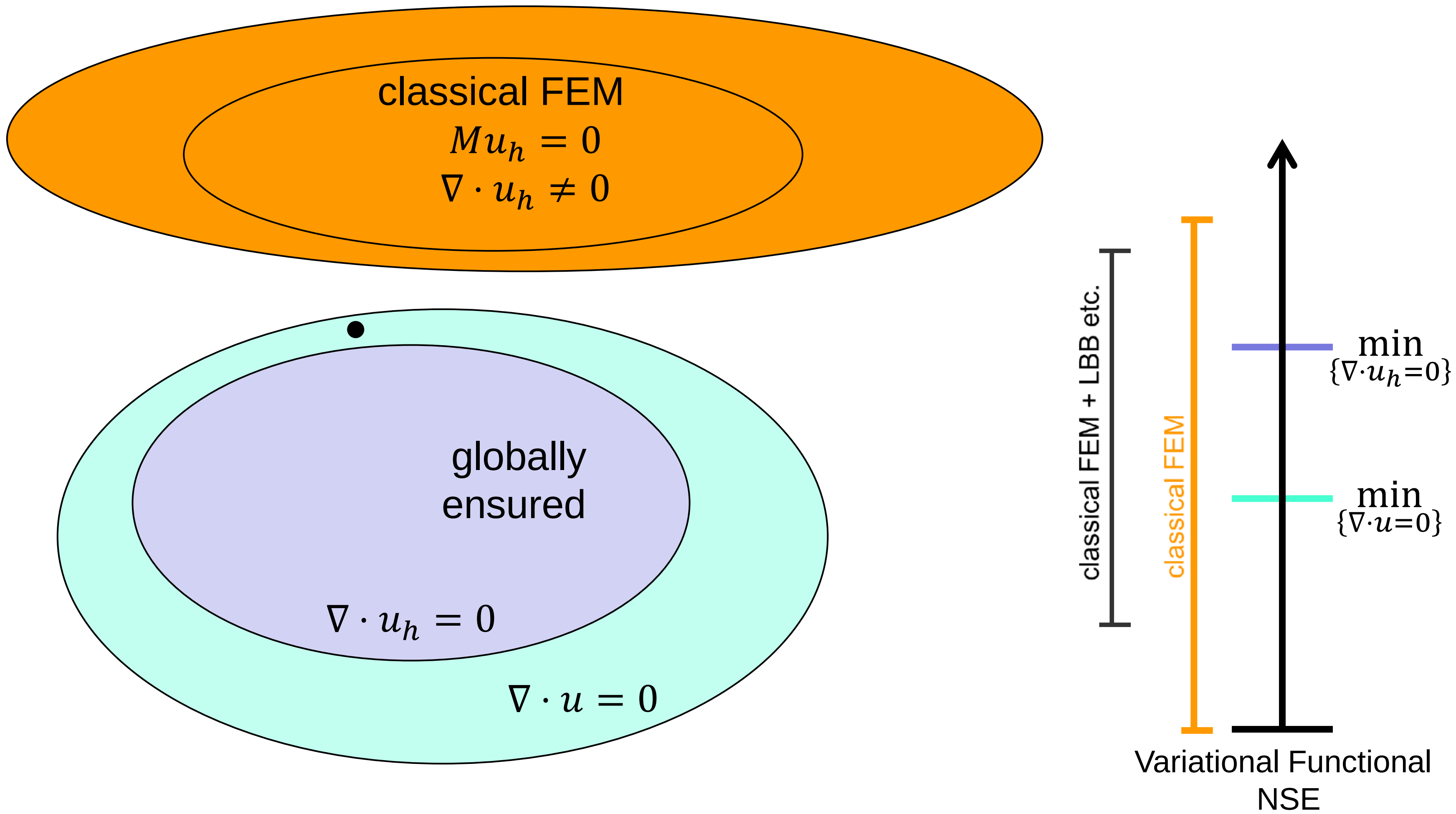

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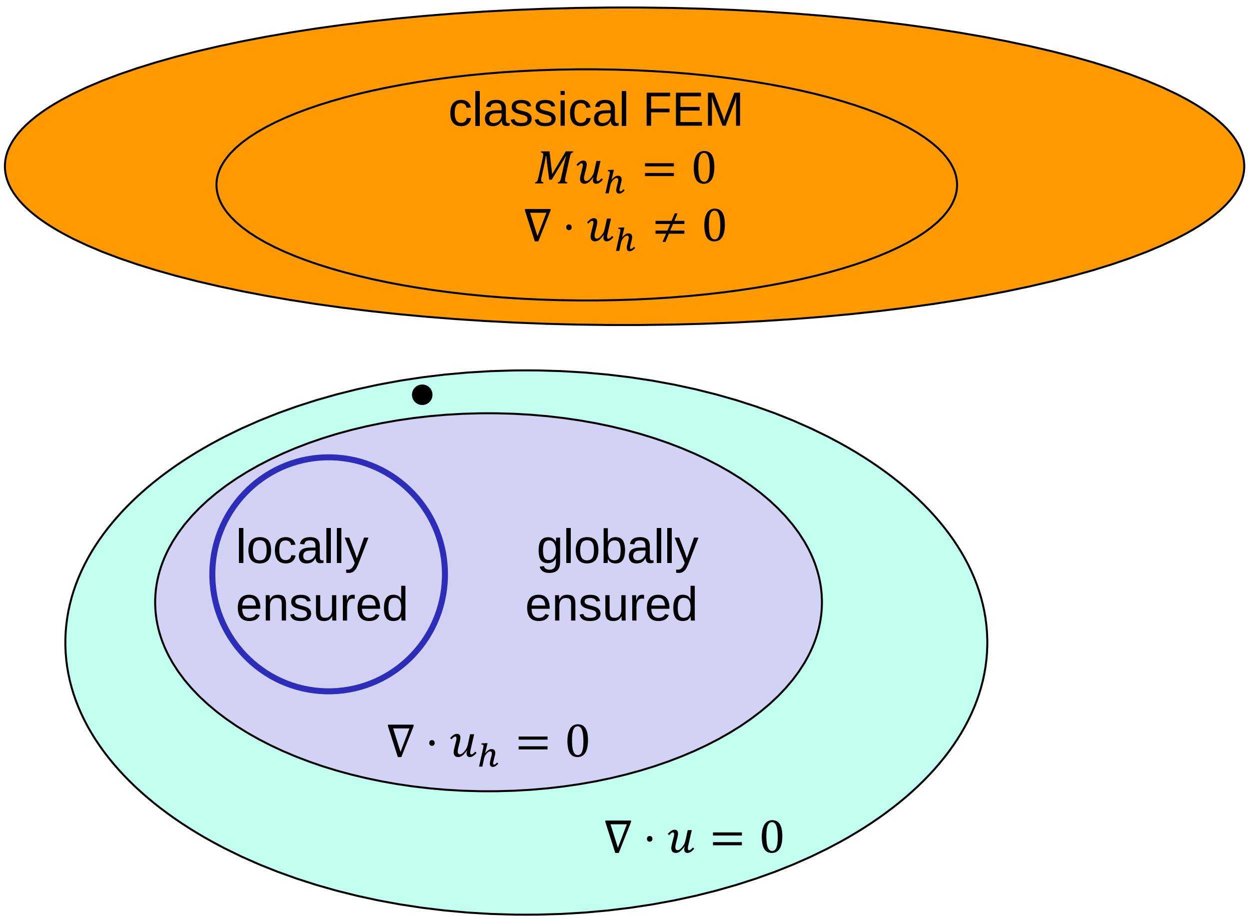
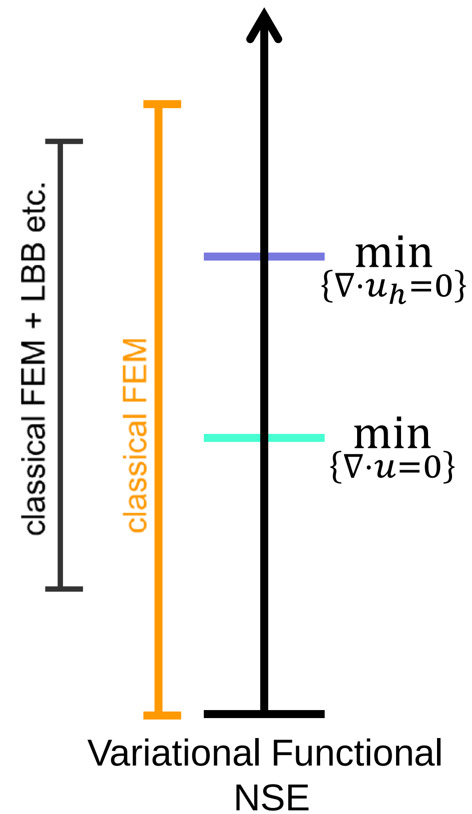




Summary of Motivations for Spatial Discretisations

symmetry properties for operators $C(u_h), D$

↓

conservation of energy **&** momentum (mimic real physics)

↓

guaranteed stable DAEs!

Summary of Motivations for Spatial Discretisations

FVM, FDM

↓ work !

symmetry properties for operators $C(u_h), D$

↓

conservation of energy **&** momentum (mimic real physics)

↓

guaranteed stable DAEs!

Summary of Motivations for Spatial Discretisations

FEM

↓ work

(locally) continuously div-free elements

↓

Correct spaces $\nabla \cdot u_h = 0$, no LBB

→ free

FVM, FDM

↓ work !

symmetry properties for operators $C(u_h), D$

↓

conservation of energy **&** momentum (mimic real physics)

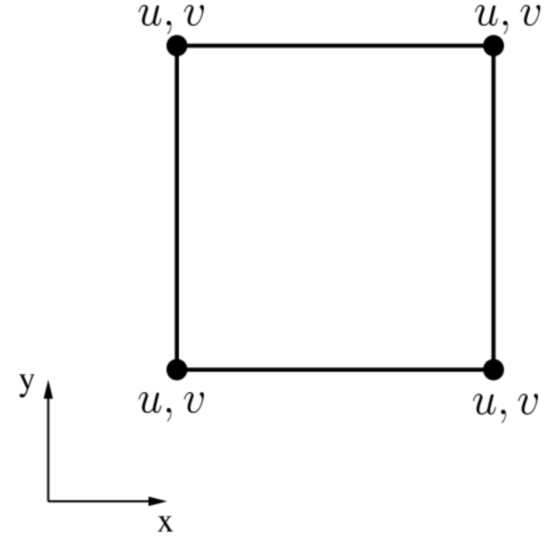
↓

guaranteed stable DAEs!

Derivation of Div-Free Ansatz Functions

- Cell-wise mass conservation:

$$0 = -u_0 + u_1 - u_2 + u_3 - v_0 - v_1 + v_2 + v_3$$



C. Blanke, Diploma Thesis, 2004

Derivation of Div-Free Ansatz Functions

- Cell-wise mass conservation:

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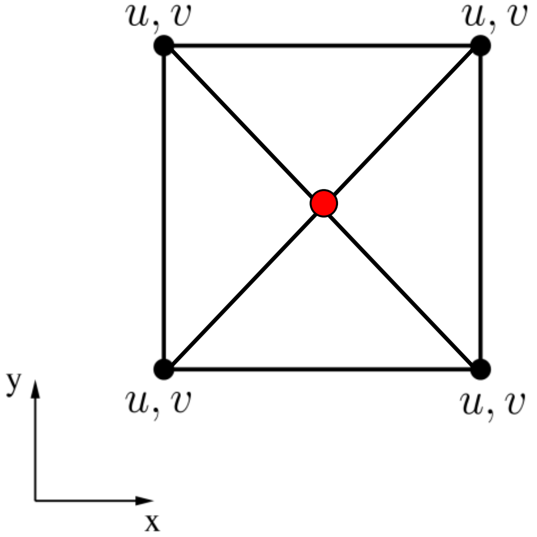
- Split cell in triangles

- Linear velocities on triangles



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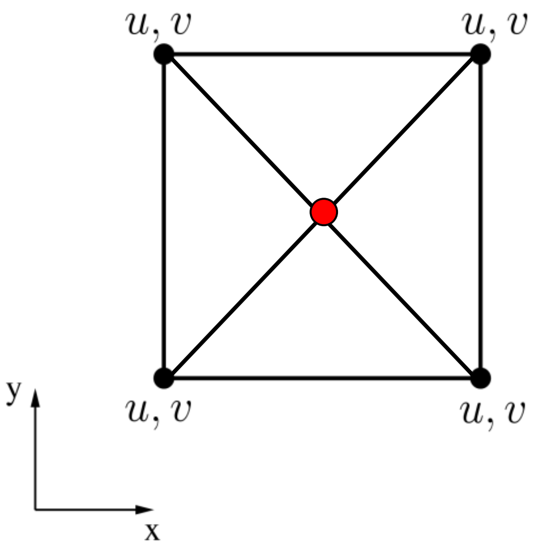
- Split cell in triangles

- Linear velocities on triangles

➔ Conditions for u, v on center node



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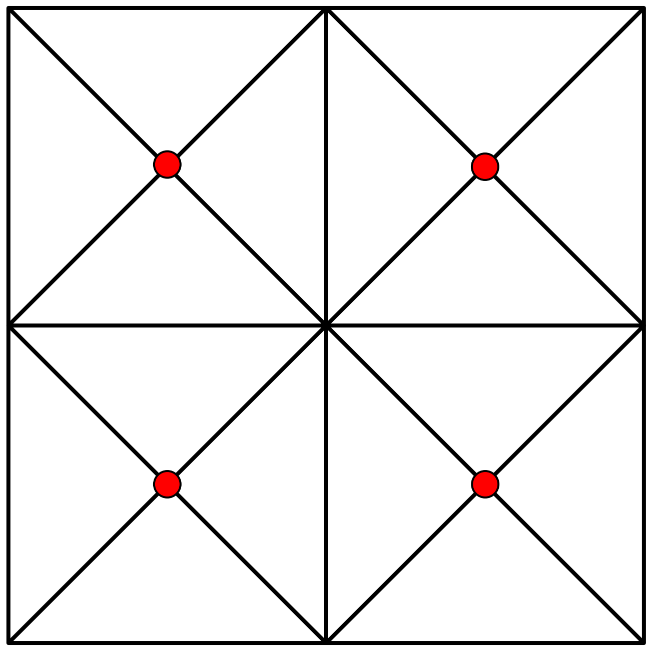
Derivation of Div-Free Ansatz Functions II



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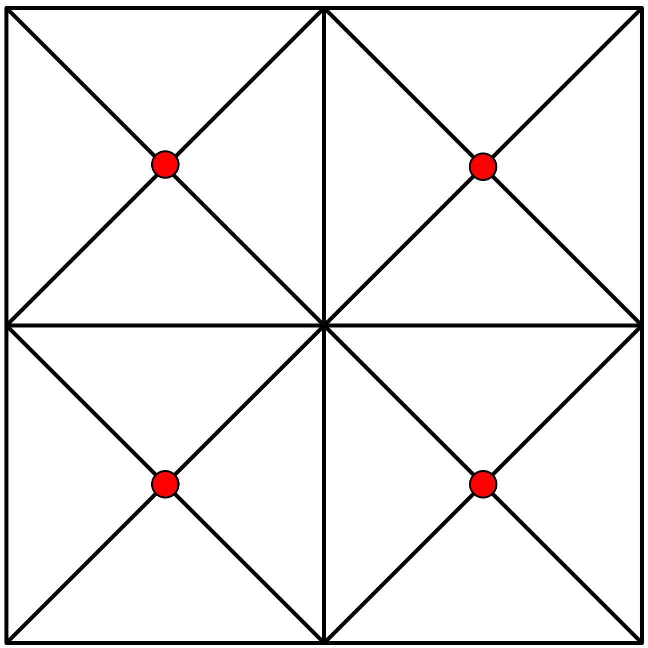
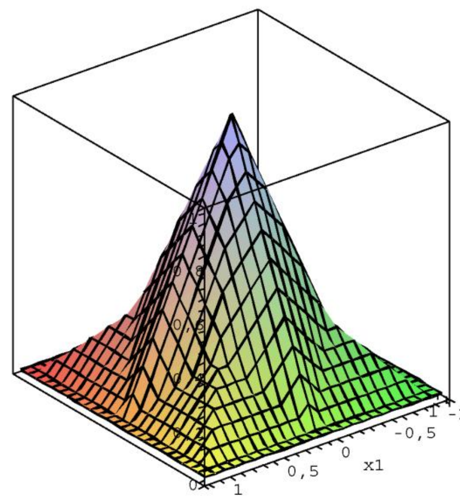
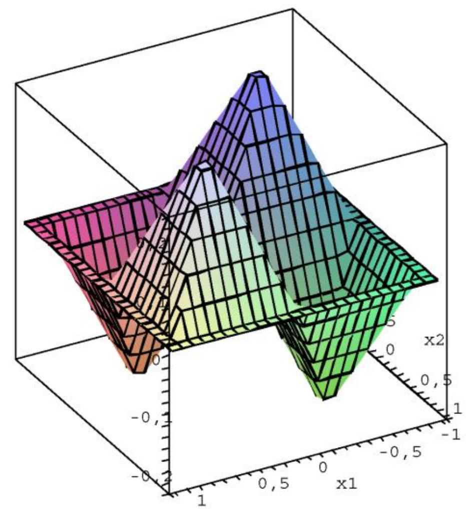




$\phi(x, y)$

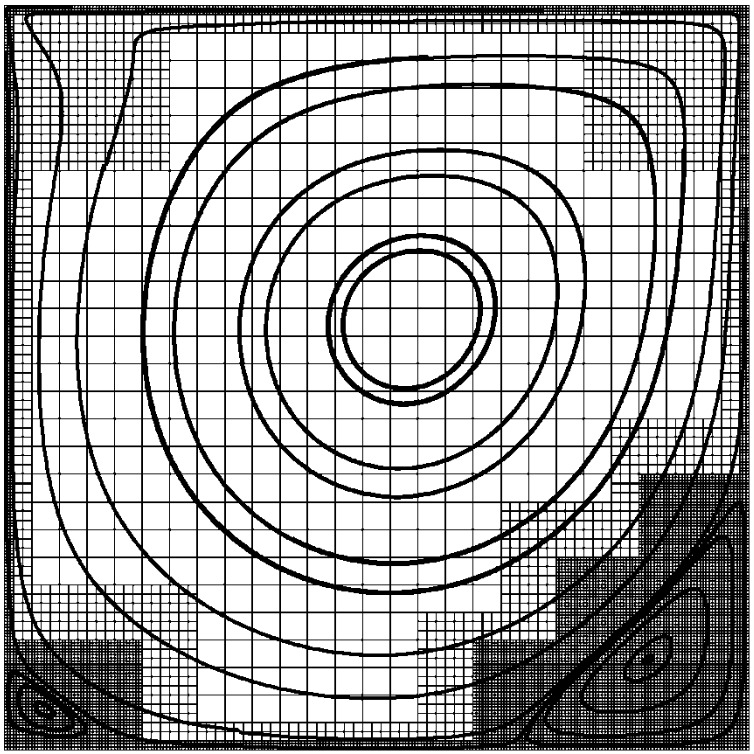


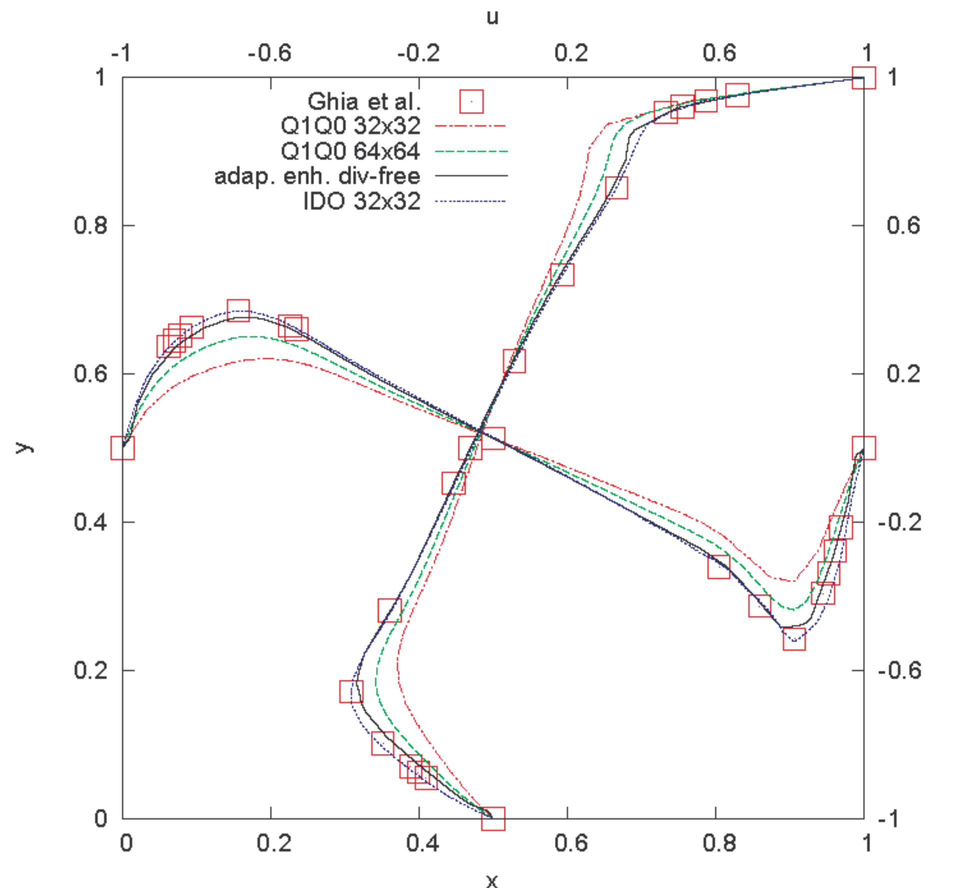
$\varphi(x, y)$

$$\boldsymbol{u}_h = \sum_{i=1}^{N} u_i \begin{pmatrix} \phi \\ \varphi \end{pmatrix} + v_i \begin{pmatrix} \varphi \\ \phi \end{pmatrix}$$

C. Blanke, Diploma Thesis, 2004

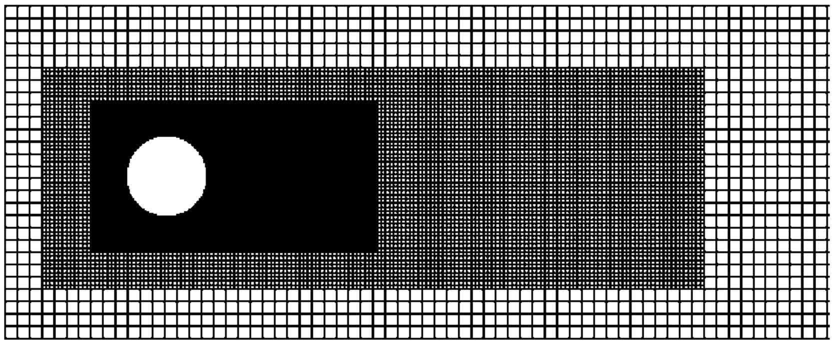
Validation – Driven Cavity



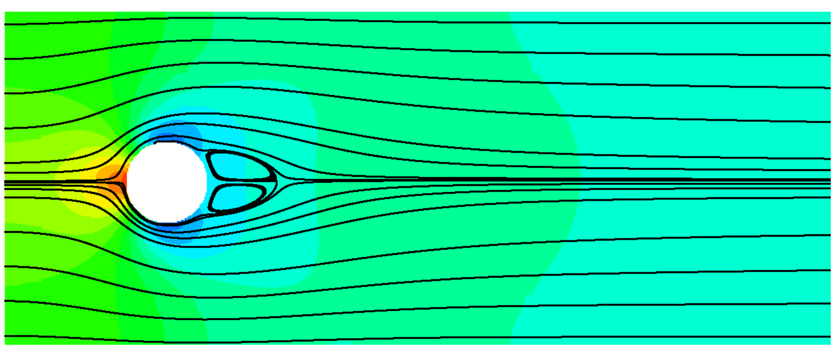


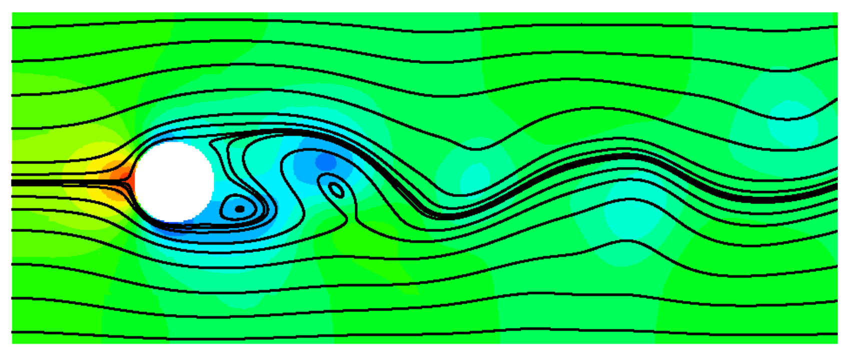
Re=1000

Validation – Flow around a Cylinder



# DoF	Re = 20		Re = 100		
	c_d	c_l	$C_{d,max}$	$C_{l,max}$	St
88,857	5.68	0.0151	3.225	0.94	0.299
ref.	5.58	0.0107	3.230	1.00	0.298





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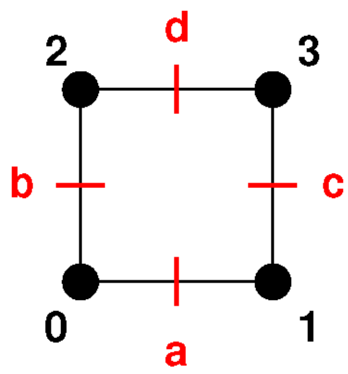
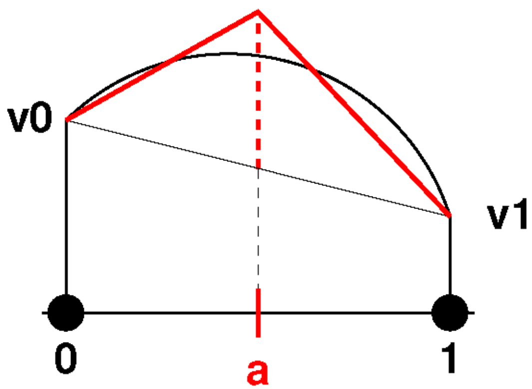
Div-free Elements – Variant 1: Enhancement

- Additional DoF on faces:
  - exact representation of fluxes on edges possible
  - no checkerboarding

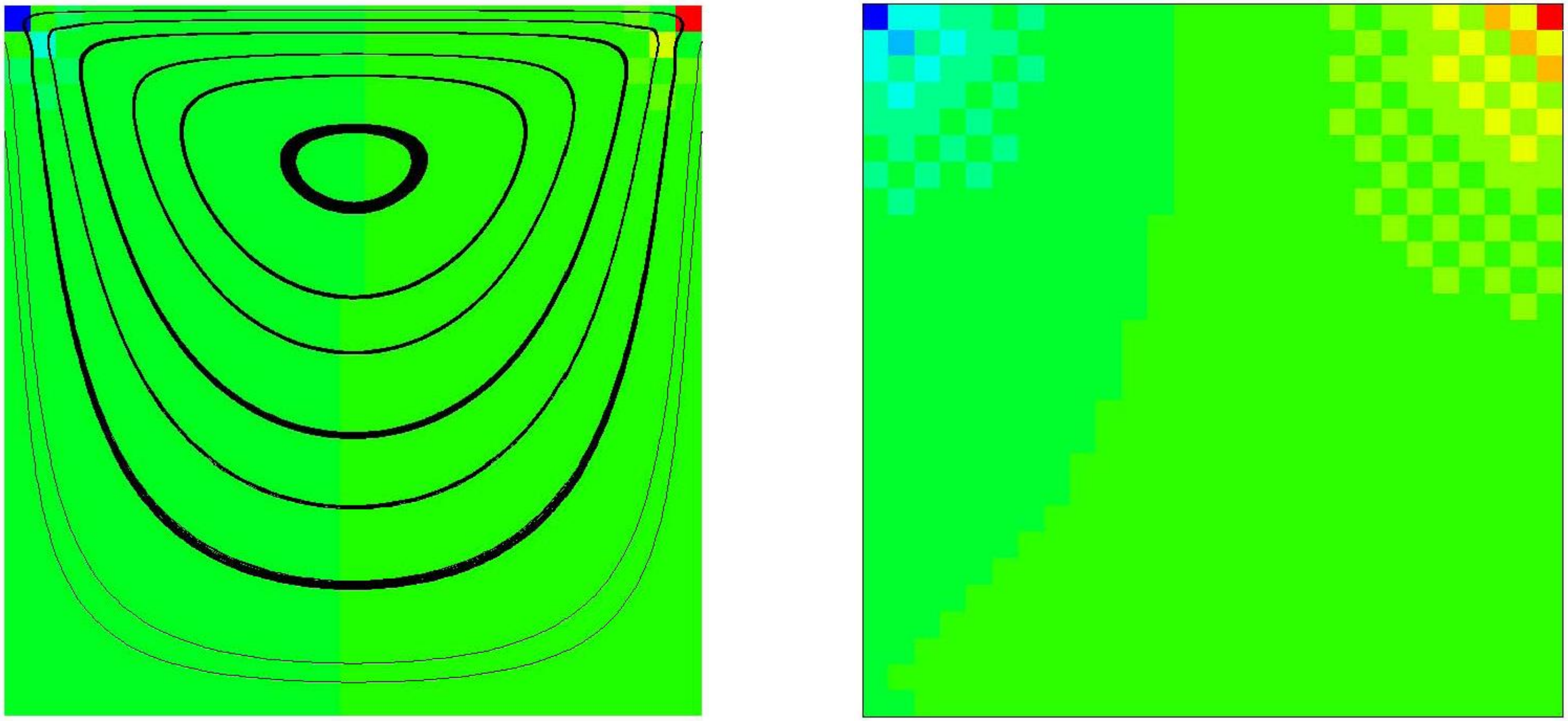




T. Weinzierl, Diploma Thesis, 2005

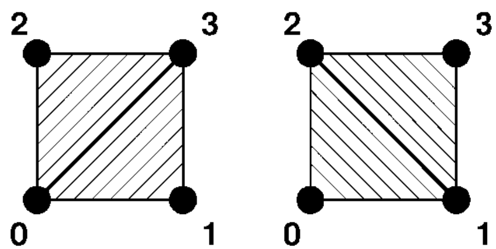
Div-free Elements – Variant 1: Enhancement

Checkerboard Driven Cavity: $u = 0{:}\,1$ in corner cells

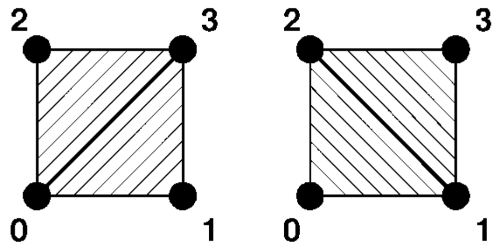


enhanced div-free (steady state) | Re=1 | Q1Q0 / div-free (step1, no convergence!)

Div-free Elements – Variant 2: Rotated Coordinates



Div-free Elements – Variant 2: Rotated Coordinates



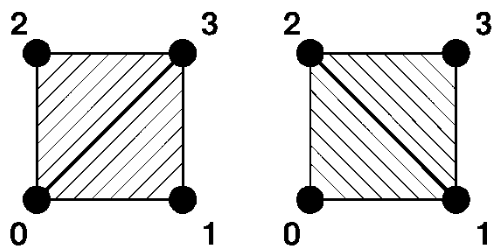
$$\frac{du}{dx} = const = u_3 - u_0$$

$$\frac{dv}{dy} = const = v_1 - v_2$$

$$\Downarrow$$

$$\nabla \cdot \mathbf{u} = u_3 - u_0 + v_1 - v2 = const = 0$$

Div-free Elements – Variant 2: Rotated Coordinates



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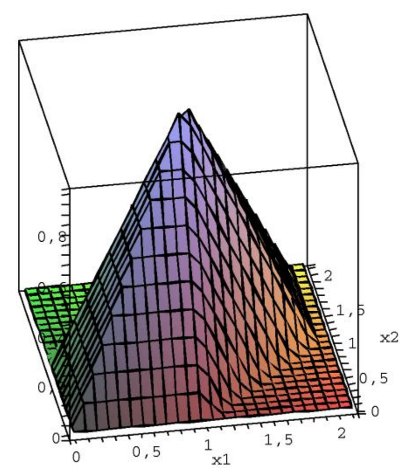
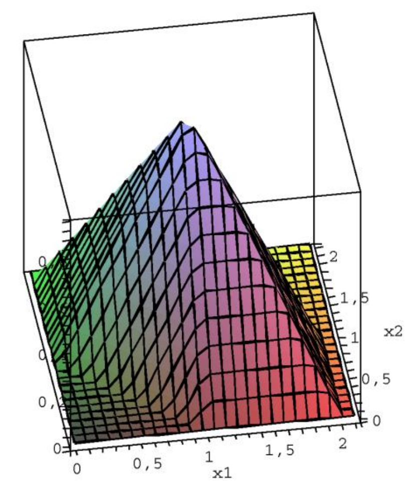


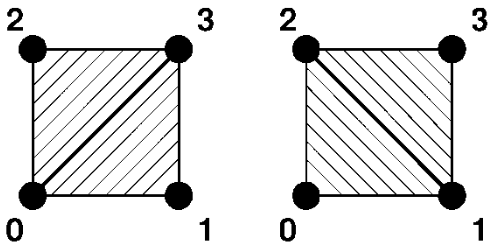
$\tilde{\phi}(x, y)$



$\tilde{\varphi}(x, y)$

$$\tilde{\boldsymbol{u}}_h = \sum_{i=1}^{N} u_i \begin{pmatrix} \tilde{\phi} \\ 0 \end{pmatrix} + v_i \begin{pmatrix} 0 \\ \tilde{\varphi} \end{pmatrix}$$

Div-free Elements – Variant 2: Rotated Coordinates



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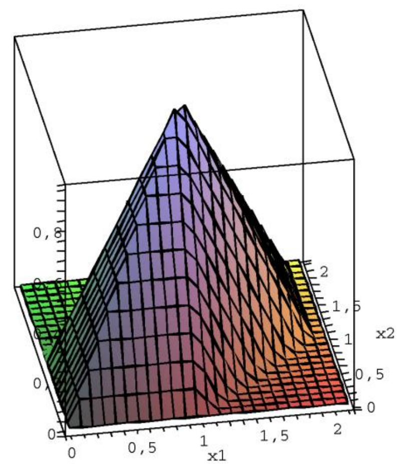

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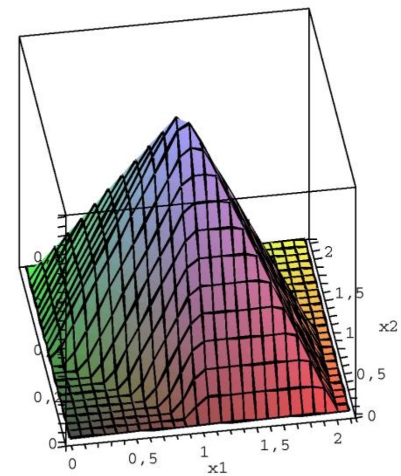


$\tilde{\varphi}(x,y)$

$$\tilde{\boldsymbol{u}}_h = \sum_{i=1}^{N} u_i \begin{pmatrix} \tilde{\phi} \\ 0 \end{pmatrix} + v_i \begin{pmatrix} 0 \\ \tilde{\varphi} \end{pmatrix}$$

Advantages:

- simpler (derivation/representation)
- performance: ~20% less runtime for evaluation of operators D and C

Summary & Outlook

Summary

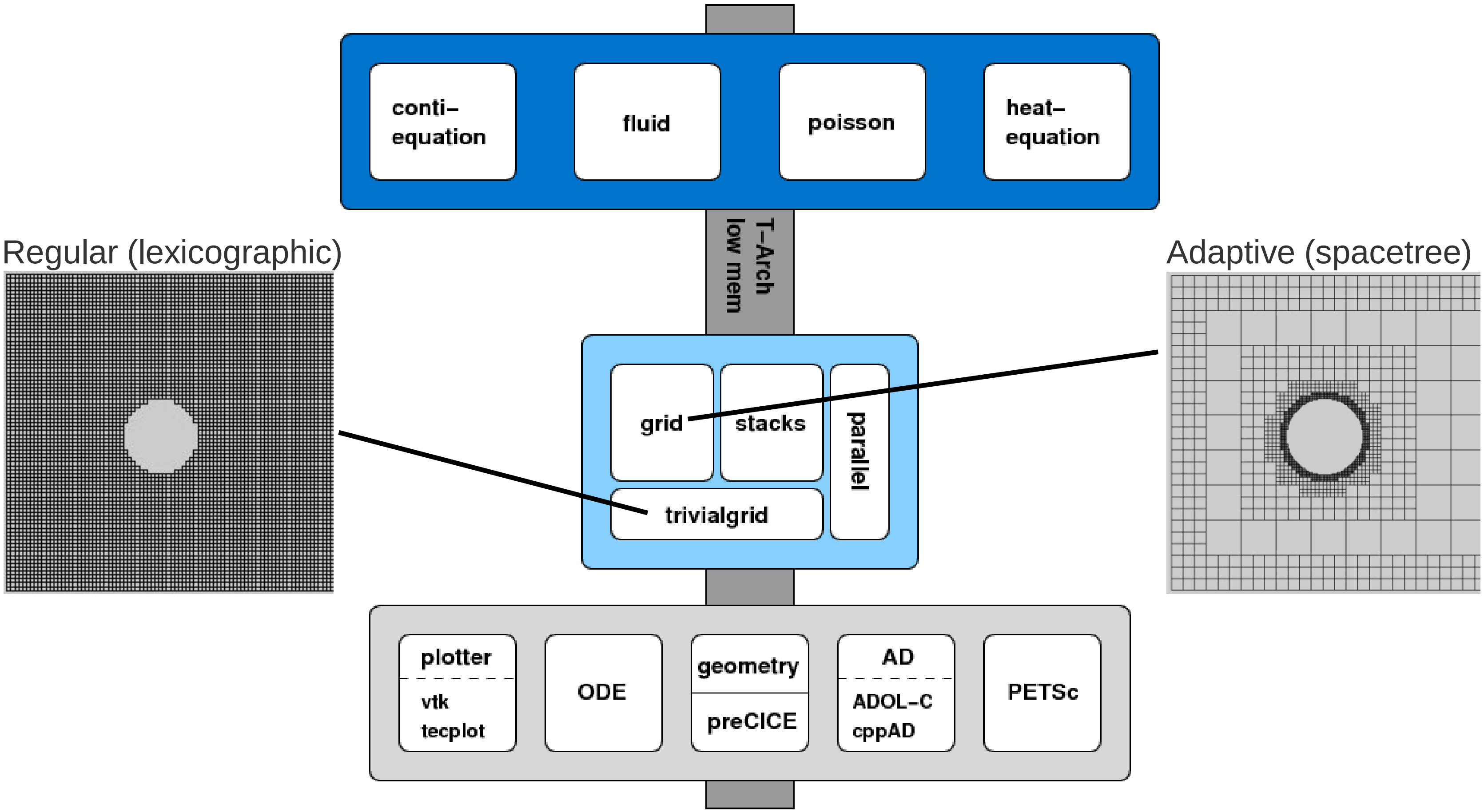
- Div-free elements
  - solenoidal on every point in cell
  - symmetry properties for discrete operators
  - conservation of discrete momentum **AND** energy
- Variants
  - Enhanced div-free elements: DoF on faces
  - 45 represenation advantageous

Outlook

- Multigrid: Peano Framework designed for hierarchical applications
- Extension of div-free elements to 3D: no direct 45 trick :-(

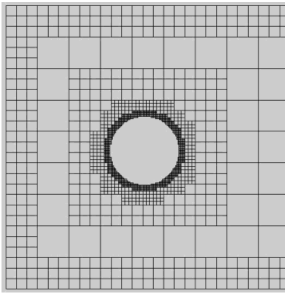
Thanks for your attention!

The PDE Framework Peano



http://www5.in.tum.de/peano/

The PDE Framework Peano

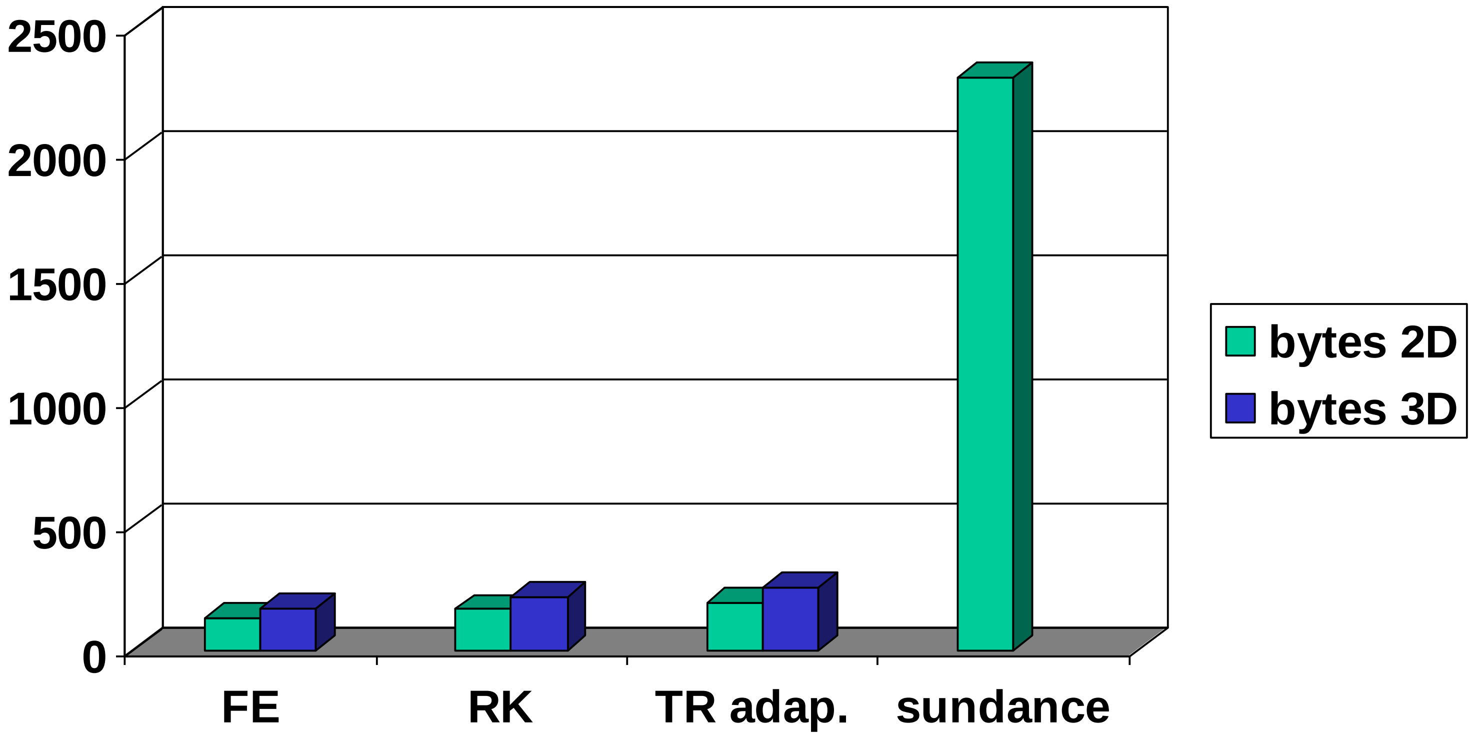


- Cartesian grids (arbitrary dimensions)
- Plug-in concept for applications
- Space-filling curves, spacetrees, and stack data structures
  - Strictly element-wise access
  - Low memory demands
  - Dynamical load balancing
  - Moving geometries, dynamical adaptivity, geometric multigrid
- Software Engineering
  - automatic tests, continuous integration, OO, design patterns, ...
- CFD component
  - Incompressible flow (FEM, IDO)
  - Explicit + implicit time-integration schemes (FE, RK4, BE, (adaptive) TR)

http://www5.in.tum.de/peano/

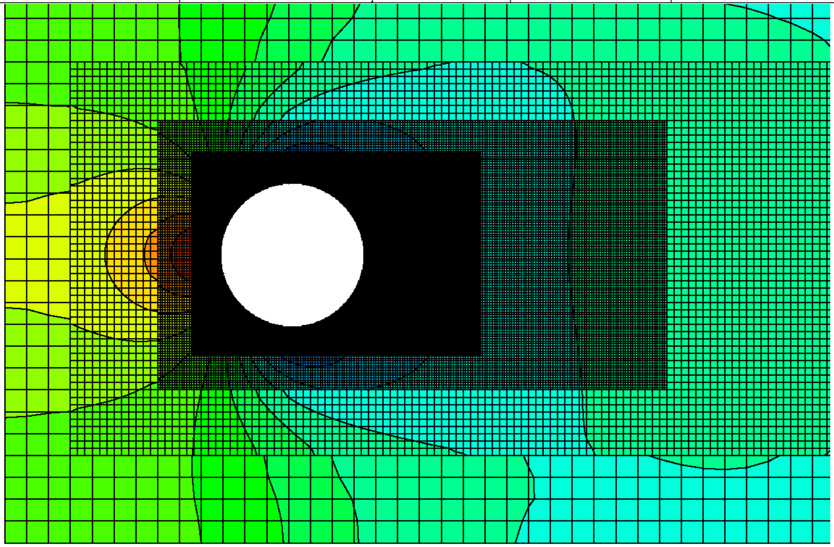
Backup I

Low memory requirements (FEM + adap.):



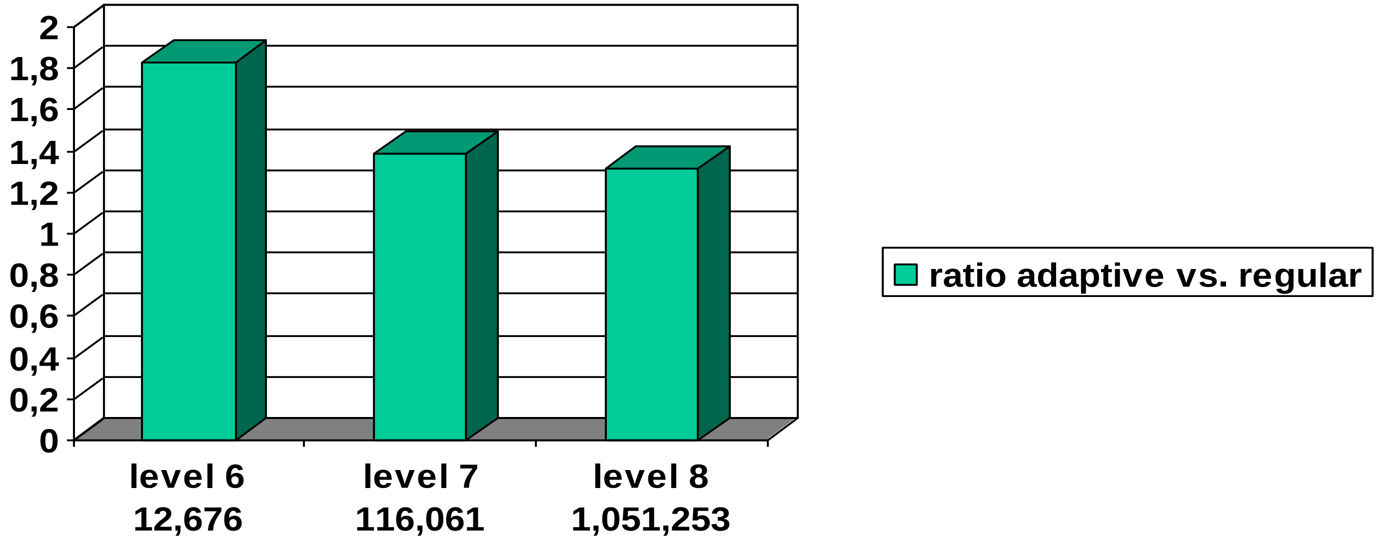
Numerical Results – FEM Q1Q0

max. level	min. level	#total DoF	c_d	c_l	CPU time per time step
8	6 (box)	88857	5.680	0.0150	0.71
9	7	125041	5.591	0.0113	1.03
9	8	1057877	5.561	0.0112	9.46
9	6 (box)	261501	5.586	0.0115	2.46
ref. data		–	5.580	0.0107	–



Numerical Results - Performance

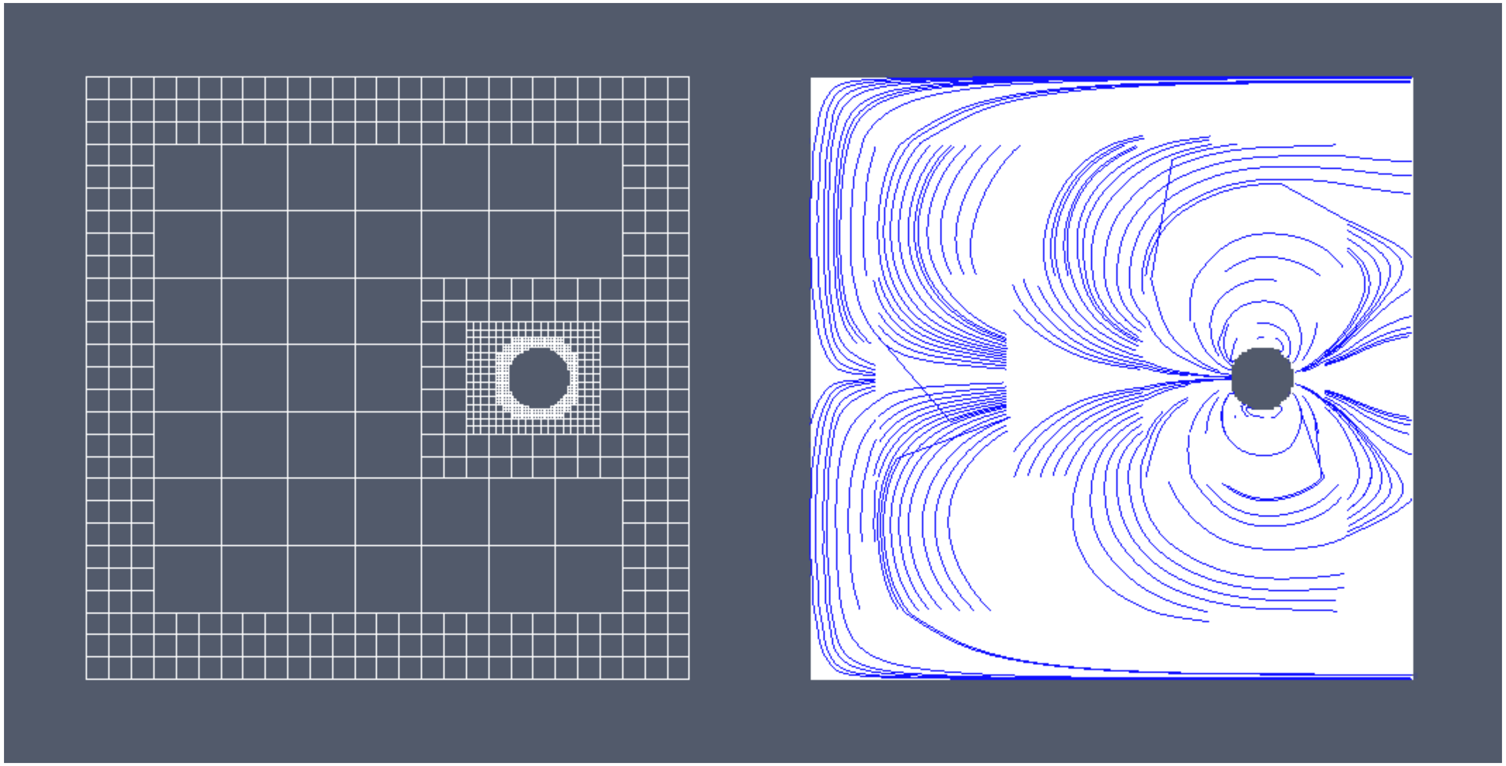
2D FEM



3D FEM Overhead adaptive vs. regular **< 3%**

2D IDO Overhead Peano vs. Aoki (regular): **1.3 – 4.4**

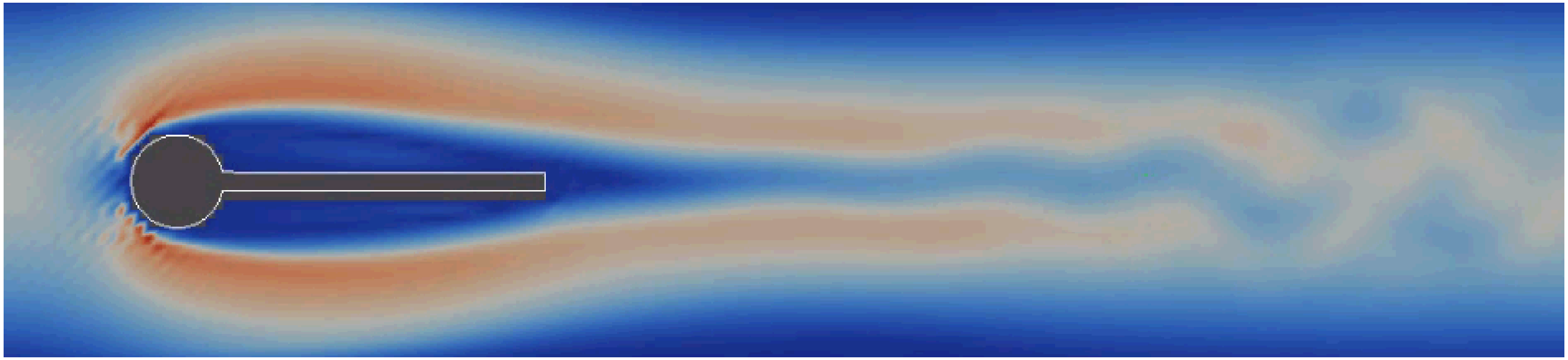
CFD Extensions



joint work with Kristof Unterweger

- Moving geometries
  - Update of data + grid (regular + adaptive)
  - Divergence correction

CFD Extensions



joint work with Janos Benk, Bernhard Gatzhammer, Miriam Mehl, Kristof Unterweger, and Tobias Weinzierl

- Moving geometries
  - Update of data + grid (regular + adaptive)
  - Divergence correction