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Good solutions of fully nonlinear parabolic equations*

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Summary. In this paper, we introduce the notion of a "good" solution of a fully nonlinear parabolic equation and show that "good" solutions are equivalent to L^p-viscosity solutions of such equations. The results here generalize the ones in [8] about "good" solutions of fully nonlinear elliptic equations. We give here an explicit construction of parabolic equations with L^p-strong solutions that approximate some nonlinear parabolic equation and its L^p-viscosity solution.

Key words: L^p-viscosity solutions, good solutions, strong solutions, fully nonlinear parabolic equations

Mathematics Subject Classification (1991): 35J60, 35J65, 49L25

1. Introduction

The notion of viscosity solutions which was introduced in 1981 by M. G. Crandall and P. L. Lions for the general Hamilton – Jacobi equations plays an important role in the theory of partial differential equations [2]. For convex Hamiltonians, the viscosity solution characterized by a semiconcave stability condition was first introduced by S. N. Kruzkov [9]. There is an enormous activity which is based on these studies. Many fundamental results about viscosity solutions for partial differential equations of first and second order have been obtained by M. G. Crandall, L. C. Evans, P. L. Lions,

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H. Ishii, R. Jensen, P. E. Souganidis, M. Bardi, G. Barles, E. N. Barron, M. Kocan, A. Swiech and so on. There results concern existence and uniqueness theorems, properties of solutions and applications.

We note that a comprehensive treatment of L^p-theory for fully nonlinear parabolic equations is presented in recent paper by M. G. Crandall, M. Kocan and A. Swiech [5].

Following [8] we introduce the notion of a good solution of a fully nonlinear parabolic equation and also prove some results that are similar to the ones for good solutions of fully nonlinear elliptic equations. The reader is referred to [1-8] for the modern theory of nonlinear elliptic and parabolic equations.

In this paper, $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is always assumed to be a bounded domain and satisfy an *uniform exterior cone condition*, $Q = \Omega \times (0;T]$, $(T > 0)$, $S(n)$ be the set of symmetric $n \times n$ real matrices. For $X \in S(n)$ we call X^+, X^- respectively the positive, negative parts of X and also define $\|X\| = \mathrm{tr}(X^+) + \mathrm{tr}(X^-)$, where $\mathrm{tr}(X)$ is the trace of X.

Let $X \in S(n)$, $0 < \lambda \leq \Lambda$. The Pucci extremal operators are defined as follows

$$\mathcal{P}^+(X) = -\lambda \mathrm{tr}(X^+) + \Lambda \mathrm{tr}(X^-), \quad \mathcal{P}^-(X) = -\Lambda \mathrm{tr}(X^+) + \lambda \mathrm{tr}(X^-).$$

Consider the nonlinear parabolic partial differential equations of the form

$$u_t + G(x,t,u,Du,D^2u) = 0 \quad \text{in} \quad Q, \tag{1.1}$$

where $G : Q \times \mathbb{R} \times \mathbb{R}^n \times S(n) \to \mathbb{R}$, $Du = (u_{x_1}, ..., u_{x_n})$, $D^2u = (u_{x_i x_j})$ correspond to the spatial gradient and Hessian matrices of u respectively.

We will always assume that $G(x,t,r,p,X)$ is continuous in (r,p,X) with modulus independent of $(x,t) \in Q$ and is jointly measurable in all variables (x,t,r,p,X).

We also require that G of (1.1) satisfy the following structure conditions

$$|G(x,t,r,p,X) - G(x,t,r,q,X)| \leq \gamma |p-q|, \tag{1.2}$$

$$\mathcal{P}^-(X-Y) \leq G(x,t,r,p,X) - G(x,t,r,p,Y) \leq \mathcal{P}^+(X-Y) \tag{1.3}$$

for all $(x,t) \in Q$, $r \in \mathbb{R}$, p, $q \in \mathbb{R}^n$, X, $Y \in S(n)$, and $\lambda \leq \Lambda$, γ are positive constants which are fixed for all time. Thus (1.3) amounts to

$$G(x,t,r,p,X) \leq G(x,t,r,p,Y) \quad \text{whenever} \quad X \geq Y. \tag{1.4}$$

As for the dependence on r, we assume that

$$(1.5) \quad \begin{cases} r \to G(x,t,r,p,X) & \text{is uniformly continuous,} \\ \text{uniformly for } (x,t) \in Q, \ |r| + |p| + \|X\| \le R \ (R > 0), \end{cases}$$

$$(1.6) \qquad r \to G(x,t,r,p,X) \quad \text{is nondecreasing.}$$

We sometimes rewrite (1.1) as follows

$$(1.7) \qquad u_t + F(x,t,u,Du,D^2u) = f(x,t)$$

by setting

$$f(x,t) = -G(x,t,0,0,0), \quad F(x,t,r,p,X) = G(x,t,r,p,X) + f(x,t),$$

so that

$$(1.8) \qquad F(x,t,0,0,0) \equiv 0.$$

Regarding the behavior of f, we require that

$$(1.9) \qquad f \in L^p(Q), \quad p > p_0,$$

where $p_0 < n+1$ is a constant such that the generalized maximum principle for parabolic equations holds for $p > p_0$.

Clearly, F of (1.7) will satisfy versions of (1.2)–(1.6) whenever G of (1.1) does, and vice versa.

The notions of solutions which are used in this work are L^p-viscosity subsolutions, L^p-viscosity supersolutions, L^p-viscosity solutions, L^p-strong solutions, pointwise a. e. solutions (see [5]).

2. Good solutions

Definition 1. *We say that the functions $G_1, G_2, \ldots, G_m, \ldots$ satisfy structure conditions uniformly in m if (1.2), (1.3), (1.5) are satisfied uniformly in m with the same fixed λ, Λ, γ and if $|G_m(x,t,0,0,0)| \le g(x,t)$ for some $g \in L^p(Q)$.*

Definition 2. *We say that $u \in C(Q)$ is a good solution of (1.1) if there exist G_m satisfying structure conditions uniformly in m and an L^p-strong solution u_m of $u_t + G_m = 0$ in Q such that $u_m \to u$ in $C(Q)$ and $G_m(x,t,r,p,X) \to G(x,t,r,p,X)$ for a. e. $(x,t) \in Q$ and all $(r,p,X) \in \mathbb{R} \times \mathbb{R}^n \times S(n)$.*

First, we are going to construct an L^p-strong solution of the Cauchy – Dirichlet problem

$$(2.1)\quad u_t - \Delta u + G(x,t,u,Du,D^2u) = 0 \quad \text{in} \quad Q, \quad u = \psi \text{ on } \partial_p Q,$$

when G is bounded. This will turn out to be an important construction in proving that an L^p-viscosity solution of (1.1) is a good solution of (1.1) as well.

Theorem 1. *Let $G : Q \times \mathbb{R} \times \mathbb{R}^n \times S(n) \to \mathbb{R}$ be measurable, bounded and satisfy (1.2), (1.4), and (1.5), $\psi \in C(\partial_p Q)$. Then the Cauchy – Dirichlet problem (2.1) has an L^p-strong solution $u \in C(\overline{Q}) \cap W^{2,1,p}_{\mathrm{loc}}(Q)$.*

Proof. We are going to solve (2.1) by the fixed point method. For any given $v \in C(\overline{Q})$ we consider the Cauchy – Dirichlet problem

$$(2.2)\quad u_t - \Delta u + G(x,t,v(x,t),Du,D^2u) = 0 \text{ in } Q, \quad u = \psi \text{ on } \partial_p Q.$$

Since the equation in (2.2) is independent of u, from [3, Theorem 4.5 and Remark 4.8] it follows that (2.2) has an L^p-viscosity solution in $C(\overline{Q})$ which we denote by Tv. By [5, Propositions 3.2 and 4.1], Tv is parabolically twice pointwise differentiable a. e., a pointwise a. e. solution and, therefore, an L^p-viscosity solution of the Cauchy – Dirichlet problem

$$(2.3)\qquad u_t - \Delta u = -g(x,t) \text{ in } Q, \quad u = \psi \text{ on } \partial_p Q,$$

where $g(x,t) = G(x,t,v(x,t),D(Tv)(x,t),D^2(Tv)(x,t))$ ($g \in L^\infty(Q)$ since G is bounded). However, it is clear that (2.3) has a unique L^p-strong solution which must coincide with Tv, and it follows that $Tv \in W^{2,1,p}_{\mathrm{loc}}(Q)$. In particular, Tv is a unique L^p-strong solution of (2.3).

On the other hand, since g is bounded it follows [3, Proposition 4.6 and Remark 4.8] that if R is large enough then T is a compact mapping in the ball with the center at 0 and the radius R in $C(\overline{Q})$. Thus, by Schaefer's fixed point theorem, T has a fixed point or (2.1) has a L^p-strong solution and the proof is complete. □

Theorem 2. (Existence of good solutions) *Let $\psi \in C(\partial_p Q)$, let F be measurable and satisfy (1.2), (1.3), (1.5), (1.6), (1.8), and $f \in L^p(Q)$. Then the Cauchy – Dirichlet problem*

$$(2.4)\qquad u_t + F(x,t,u,Du,D^2u) = f(x,t) \text{ in } Q, \quad u = \psi \text{ on } \partial_p Q$$

has a good solution.

Proof. Noting that under the same conditions we know that (2.4) also has an L^p-viscosity solution. We will use Theorem 1 to construct good solutions of (2.4). For this we rewrite the differential equation in (2.4) as follows

$$u_t - \lambda \Delta u + G(x, t, u, Du, D^2 u) = 0,$$

where

$$G(x, t, r, p, X) = \lambda \mathrm{tr}(X) + F(x, t, r, p, X) - f(x, t).$$

Clearly, G satisfies (1.4).

Without loss of generality we may replace (2.4) by

$$u_t - \lambda \Delta u + G(x, t, u, Du, D^2 u) = 0 \text{ in } Q, \quad u = \psi \text{ on } \partial_p Q.$$

For $m = 1, 2, \ldots$, consider truncating functions

$$\chi_m(r) = \max(\min(r, m), -m)$$

and an approximating Cauchy – Dirichlet problem

(2.5) $u_t - \lambda \Delta u + \chi_m(G(x, t, u, Du, D^2 u)) = 0 \text{ in } Q, \quad u = \psi \text{ on } \partial_p Q.$

Since $\chi_m(G(x, t, u, Du, D^2 u))$ satisfies the condition of Theorem 1 and G does (1.4), by similar argument, we assert that (2.5) has a unique L^p-strong solution $u_m \in C(\overline{Q}) \cap W^{2,1,p}_{\mathrm{loc}}(Q)$. The family of equations satisfies structure conditions uniformly in m, and by [3, Proposition 4.6 and Remark 4.8] the set $\{u_m\}$ is precompact in $C(\overline{Q})$. Therefore, passing to a subsequence if necessary, we can assume that $u_m \to u$ in $C(\overline{Q})$. The function $u \in C(\overline{Q})$ is a desired good solution since the approximations in (2.5) obviously converge to G a. e. and the proof is complete. □

Next, we consider the Cauchy – Dirichlet problem

(2.6) $$u_t + F(x, t, Du, D^2 u) = f(x, t) \text{ in } Q, \quad u = \psi \text{ on } \partial_p Q.$$

We will prove that every L^p-viscosity solution of (2.6) is a good solution. The fact that good solutions are L^p-viscosity solutions is obvious from the definition of good solution. First, we remind the *sup-convolution* w^ε and *inf-convolution* w_ε of a function $w : \overline{Q} \to \mathbb{R}$, $\varepsilon > 0$, as

$$w^\varepsilon(x, t) = \sup_{y \in \Omega} \left(w(y, t) - \frac{1}{2\varepsilon} |x - y|^2 \right),$$

$$w_\varepsilon(x, t) = \inf_{y \in \Omega} \left(w(y, t) + \frac{1}{2\varepsilon} |x - y|^2 \right), \quad (x, t) \in Q.$$

Now for $\varepsilon > 0$, $\delta > 0$, we put

$$w^{\varepsilon, \delta} = (w^{\varepsilon + \delta})_\delta, \quad w_{\varepsilon, \delta} = (w_{\varepsilon + \delta})^\delta.$$

Theorem 3. *Let F be measurable and satisfy (1.2), (1.3), (1.8), let f satisfy (1.9), $\psi \in C(\partial_p Q)$. Then every L^p-viscosity solution of (2.6) is a good solution, i. e. there are a sequence of operators F_m independent of u, satisfying (1.2), (1.3), and (1.8), a sequence $f_m \in L^p(Q)$ and a sequence $u_m \in C(\overline{Q}) \cap W^{2,1,p}_{loc}(Q)$ of L^p-strong solutions of*

$$(u_m)_t + F_m(x,t,Du_m,D^2u_m) = f_m(x,t) \quad in \quad Q$$

such that

$$u_m \to u \quad in \quad C(\overline{Q}), \tag{2.7}$$

$$F_m(x,t,p,X) \to F(x,t,p,X) \tag{2.8}$$

for a. e. $(x,t) \in Q$ and for all $(p,X) \in \mathbb{R}^n \times S(n)$, and

$$f_m \to f \quad in \quad L^p(Q). \tag{2.9}$$

Proof. We will use the method proposed in [8] in order to prove Theorem 3. Indeed we repeat all steps of the proof of Theorem 3.1 in [8] for elliptic equations. Assume that u is an L^p-viscosity solution of (2.6).

Step 0. Firstly we fix a countable, dense in $\mathbb{R}^n \times S(n)$ sequence $(p_i, X_i) \in \mathbb{R}^n \times S(n)$, $i = 1, 2, \ldots$.

Step 1. Choose a sequence $\tilde{f}_m \in C(Q) \cap L^\infty(Q)$ such that $\delta_m = \|f - \tilde{f}_m\|_{L^p(Q)} \to 0$. By [5, Theorem 2.8], the following problem

$$W_t + \mathcal{P}^-(D^2W) - \gamma|DW| = f - \tilde{f}_m \text{ in } Q, \quad W = 0 \text{ on } \partial_p Q$$

has a unique L^p-strong solution W and $\|W\|_{L^\infty(Q)} \le C\delta_m$. Let $\overline{u}_m = u - W$. Hence $\overline{u}_m$ is an L^p-viscosity solution of

$$(\overline{u}_m)_t + F(x,t,D\overline{u}_m,D^2\overline{u}_m) \le \tilde{f}_m(x,t) \quad \text{in} \quad Q \tag{2.10}$$

and

$$\|u - \overline{u}_m\|_{L^\infty(Q)} = \|W\|_{L^\infty(Q)} \le C\delta_m. \tag{2.11}$$

Similarly, solving

$$W_t + \mathcal{P}^+(D^2W) + \gamma|DW| = f - \tilde{f}_m \text{ in } Q, \quad W = 0 \text{ on } \partial_p Q$$

and setting $\underline{u}_m = u - W$, we conclude that $\underline{u}_m$ is an L^p-viscosity solution of

$$(\underline{u}_m)_t + F(x,t,D\underline{u}_m,D^2\underline{u}_m) \ge \tilde{f}_m(x,t) \quad \text{in} \quad Q$$

and

$$\|u - \underline{u}_m\|_{L^\infty(Q)} \leq C\delta_m. \tag{2.12}$$

Moreover, $\overline{u}_m = u = \psi = \underline{u}_m$ on $\partial_p Q$.

Step 2. Let $\Omega_m \Subset \Omega$ be a subdomain of Ω with a smooth boundary and such that

$$\text{if} \quad x \in \Omega \setminus \Omega_m \quad \text{then} \quad d(x, \partial\Omega) \leq \frac{1}{m}. \tag{2.13}$$

Let $Q_m = \Omega_m \times (0; T]$. It follows from the proof of Lemma 1.2 in [6] and [8] that, for every sufficiently small $\varepsilon > 0$, one can choose a suitable $\delta = \delta(\varepsilon)$ such that $u_m^+ = (\overline{u}_m)^{\varepsilon,\delta} \in W^{2,1,\infty}(\overline{Q}_m)$ and it is an L^p-strong solution of a perturbed version of (2.10), namely, for a. e. $(x,t) \in Q_m$,

$$(u_m^+)_t + \overline{F}_m(x, t, Du_m^+(x,t), D^2u_m^+(x,t)) \leq \overline{f}_m(x,t), \tag{2.14}$$

where

$$\overline{F}_m(x,t,p,X) = F(T_m^+ x, t, p, X), \quad \overline{f}_m(x,t) = \tilde{f}_m(T_m^+ x, t) \tag{2.15}$$

with $T_m^+ x = x + \varepsilon Du_m^+$.

Similarly, we conclude that $u_m^- = (\underline{u}_m)_{\varepsilon,\delta} \in W^{2,1,\infty}(\overline{Q}_m)$, and

$$(u_m^-)_t + \underline{F}_m(x, t, Du_m^-(x,t), D^2u_m^-(x,t)) \geq \underline{f}_m(x,t) \tag{2.16}$$

for a. e. $(x,t) \in Q_m$, where

$$\underline{F}_m(x,t,p,X) = F(T_m^- x, t, p, X), \quad \underline{f}_m(x,t) = \tilde{f}_m(T_m^- x, t) \tag{2.17}$$

with $T_m^- x = x - \varepsilon Du_m^-$.

Also there is $\beta_m > 0$ independent of ε such that

$$DT_m^\pm \geq \beta_m I \text{ for a. e. } x \in \Omega_m, \tag{2.18}$$

where I is the $(n \times n)$ identity matrix.

It follows from (2.18) that $\overline{F}_m$ and $\underline{F}_m$ are measurable. Moreover, decreasing ε if necessary, we can achieve that

$$\|\overline{f}_m - \tilde{f}_m\|_{L^p(Q_m)} \leq \frac{1}{m}, \quad \|\underline{f}_m - \tilde{f}_m\|_{L^p(Q_m)} \leq \frac{1}{m} \tag{2.19}$$

and

$$\int_{Q_m} |F(T_m^\pm x, t, p_i, X_i) - F(x, t, p_i, X_i)| dxdt \leq \frac{1}{m}, \tag{2.20}$$

$$i = 1, 2, \ldots, m.$$

Further, without loss of generality we may also assume that

$$\|\overline{u}_m - u_m^+\|_{L^\infty(Q_m)} \le C\delta_m, \quad \|\underline{u}_m - u_m^-\|_{L^\infty(Q_m)} \le C\delta_m. \tag{2.21}$$

Finally, redefining $u_m^- = u_m^- - 3C\delta_m$ and $u_m^+ = u_m^+ + 3C\delta_m$ we see that (2.14) and (2.16) still hold, while by (2.11), (2.12), and (2.21)

$$u - 5C\delta_m \le u_m^- \le u - C\delta_m, \; u + C\delta_m \le u_m^+ \le u + 5C\delta_m \text{ on } \overline{Q}_m. \tag{2.22}$$

Step 3. We now establish some limiting properties of the approximations constructed in Step 2, which will be needed later in Step 6.

From (2.20), for every (p_i, X_i) and $Q' \Subset Q$, we have

$$\int_{Q'} |F(T_m^\pm x, t, p_i, X_i) - F(x, t, p_i, X_i)| dx dt \to 0 \text{ as } m \to \infty.$$

Thus, there exists a subsequence m_k and a null set $N \subset Q$ such that for all $(x,t) \in Q \setminus N$ and i

$$F(T_{m_k}^\pm x, t, p_i, X_i) \to F(x, t, p_i, X_i) \text{ as } k \to \infty.$$

Using structure conditions (1.2) and (1.3) we can generalize this convergence, namely, for all $(x,t) \in Q \setminus N$ and $(p, X) \in \mathbb{R}^n \times S(n)$

$$F(T_{m_k}^\pm x, t, p, X) \to F(x, t, p, X) \text{ as } k \to \infty. \tag{2.23}$$

Step 4. Next choose a constant $M_m \ge m$ such that for a. e. $(x,t) \in \overline{Q}_m$

$$\begin{aligned} &|\overline{F}_m(x, t, Du_m^+(x,t), D^2u_m^+(x,t)) + \lambda \Delta u_m^+(x,t)| \le M_m, \\ &|\underline{F}_m(x, t, Du_m^-(x,t), D^2u_m^-(x,t)) + \lambda \Delta u_m^-(x,t)| \le M_m. \end{aligned} \tag{2.24}$$

Let

$$\overline{G}_m(x,t,p,X) = -\lambda \operatorname{tr}(X) + \chi_{M_m}(\overline{F}_m(x,t,p,X) + \lambda \operatorname{tr}(X)).$$

It follows from (2.14) and (2.24) that

$$(u_m^+)_t + \overline{G}_m(x, t, Du_m^+(x,t), D^2u_m^+(x,t)) \le \overline{f}_m(x,t)$$

for a. e. $(x,t) \in Q_m$. Similarly,

$$(u_m^-)_t + \underline{G}_m(x, t, Du_m^-(x,t), D^2u_m^-(x,t)) \ge \underline{f}_m(x,t) \tag{2.25}$$

for a. e. $(x,t) \in Q_m$, where

$$\underline{G}_m(x,t,p,X) = -\lambda \operatorname{tr}(X) + \chi_{M_m}(\underline{F}_m(x,t,p,X) + \lambda \operatorname{tr}(X)).$$

Let $\mu : \overline{Q}_m \times \mathbb{R} \to [0;1]$ be a continuous function such that

$$\mu(x,t,s) = \begin{cases} 0 & \text{if } u_m^+(x,t) \leq s \\ 1 & \text{if } u_m^-(x,t) \geq s. \end{cases}$$

This can be done due to (2.22). Define

$$H_m(x,t,r,p,X) = \mu(x,t,r)\overline{G}_m(x,t,p,X)$$
$$+(1-\mu(x,t,r))\underline{G}_m(x,t,p,X) = -\lambda \mathrm{tr}(X)$$
$$+\mu(x,t,r)\chi_{M_m}(\overline{F}_m(x,t,p,X) + \lambda \mathrm{tr}(X))$$
$$+(1-\mu(x,t,r))\chi_{M_m}(\underline{F}_m(x,t,p,X) + \lambda \mathrm{tr}(X)),$$

$$h_m(x,t,r) = \mu(x,t,r)\overline{f}_m(x,t) + (1-\mu(x,t,r))\underline{f}_m(x,t) \tag{2.26}$$

and consider the Cauchy – Dirichlet problem

$$v_t + H_m(x,t,v,Dv,D^2v) = h_m(x,t,v) \text{ in } Q_m,\ v = u \text{ on } \partial_p Q_m. \tag{2.27}$$

All assumptions of Theorem 1 are satisfied and thus (2.27) has an L^p-strong solution $v_m \in C(\overline{Q}_m) \cap W^{2,1,p}_{\mathrm{loc}}(Q_m)$. Now we define

$$G_m(x,t,p,X) = H_m(x,t,v_m(x,t),p,X),$$

$$g_m(x,t) = h_m(x,t,v_m(x,t))$$

so that v_m solves

$$(v_m)_t + G_m(x,t,Dv_m(x,t),D^2v_m(x,t)) = g_m(x,t),$$

for a. e. $(x,t) \in Q_m$, $v_m = u$ on $\partial_p Q_m$. Observe that by construction

$$\|g_m\|_{L^\infty(Q_m)} \leq \|\tilde{f}_m\|_{L^\infty(Q)},\ |\lambda \mathrm{tr}(X) + G_m(x,t,p,X)| \leq M_m. \tag{2.28}$$

Moreover, from (2.19) and (2.26) we have

$$\|g_m - \tilde{f}_m\|_{L^p(Q_m)} \leq \|\overline{f}_m - \tilde{f}_m\|_{L^p(Q_m)} + \|\underline{f}_m - \tilde{f}_m\|_{L^p(Q_m)} \leq \frac{2}{m}. \tag{2.29}$$

Next we claim that

$$v_m \leq u_m^+ + 8C\delta_m \text{ on } \overline{Q}_m. \tag{2.30}$$

To show (2.30) we consider $Q_m^+ = \{(x,t) \in Q_m : v_m(x,t) > u_m^+(x,t)\}$. Then $\mu(x,t,v_m(x,t)) \equiv 0$ on Q_m^+. Therefore,

$$g_m(x,t) = \underline{f}_m(x,t) \text{ and } G_m(x,t,p,X) = \underline{G}_m(x,t,p,X)$$

for $(x,t) \in Q_m^+$. It follows that v_m is an L^p-strong solution of $u_t + \underline{G}_m = \underline{f}_m$ on Q_m^+. Since by (2.25) u_m^- is a supersolution of the same equation, by the minimum principle and (2.22) we have

$$\inf_{Q_m^+}(u_m^- - v_m) \geq \inf_{\partial_p Q_m^+}(u_m^- - v_m)$$

$$\geq \min\big(\inf_{\partial_p Q_m^+ \setminus \partial_p Q_m}(u_m^- - u_m^+); \inf_{\partial_p Q_m}(u_m^- - u)\big) \geq -10C\delta_m.$$

Use of (2.22) again yields (2.30).

A symmetric argument shows that

$$v_m \geq u_m^- - 8C\delta_m \text{ on } \overline{Q}_m. \tag{2.31}$$

Summarizing (2.30), (2.31), and (2.22) together gives

$$u - 13C\delta_m \leq v_m \leq u + 13C\delta_m \text{ on } \overline{Q}_m. \tag{2.32}$$

Step 5. We now extend g_m and G_m to the whole Q as follows

$$f_m(x,t) = \begin{cases} g_m(x,t) & \text{for } (x,t) \in Q_m \\ 0 & \text{for } (x,t) \in Q \setminus Q_m, \end{cases}$$

$$F_m(x,t,p,X) = \begin{cases} G_m(x,t,p,X) & \text{for } (x,t) \in Q_m \\ -\lambda \mathrm{tr}(X) & \text{for } (x,t) \in Q \setminus Q_m. \end{cases}$$

Taking into account (2.28), by Theorem 1, the Cauchy – Dirichlet problem

$$u_t + F_m(x,t,Du,D^2u) = f_m \text{ in } Q, \quad u = \psi \text{ on } \partial_p Q$$

has a unique L^p-strong solution $u_m \in C(\overline{Q}) \cap W^{2,1,p}_{\mathrm{loc}}(Q)$. Moreover, it is similar to Remark 4.3 in [3], there exists a modulus of continuity ρ determined only by the parameters of the cone condition λ, Λ, n, p, γ, $\|f_m\|_{L^p(Q)}$, the modulus of continuity of ψ, $\mathrm{diam}(Q)$ such that

$$|u_m(x,t) - \psi(y,s)| \leq \rho(|x-y| + |t-s|) \tag{2.33}$$

$(x,t) \in Q$, $(y,s) \in \partial_p Q$.

Step 6. Denote by ρ_u the modulus of continuity of u on $\overline{Q}$. From (2.13) and (2.33) we conclude that

$$|u_m(x,t) - u(x,t)| \leq \rho(\frac{1}{m}) + \rho_u(\frac{1}{m}),\ (x,t) \in Q \setminus Q_m. \tag{2.34}$$

Since both u_m and v_m solve the same equation $u_t + G_m = g_m$ in Q_m and $v_m = u$ on $\partial_p Q_m$, by the maximum principle and (2.34) we have

$$\sup_{(x,t)\in Q_m} |u_m - v_m| \leq \sup_{(x,t)\in \partial_p Q_m} |u_m - u| \leq \rho(\frac{1}{m}) + \rho_u(\frac{1}{m}).$$

By (2.32) this yields

$$\sup_{(x,t)\in Q_m} |u_m - u| \le \rho(\frac{1}{m}) + \rho_u(\frac{1}{m}) + 13C\delta_m.$$

Using (2.34) again we conclude that

$$\sup_{(x,t)\in Q} |u_m - u| \le \rho(\frac{1}{m}) + \rho_u(\frac{1}{m}) + 13C\delta_m$$

and therefore (2.7) follows. From (2.29)

$$\|f_m - f\|_{L^p(Q)} \le \|f\|_{L^p(Q\setminus Q_m)} + \|g_m - \tilde{f}_m\|_{L^p(Q_m)}$$
$$+\|f - \tilde{f}_m\|_{L^p(Q)} \le \|f\|_{L^p(Q\setminus Q_m)} + \frac{2}{m} + \delta_m$$

which, together with (2.13), establishes (2.9) along a subsequence.

Finally we show that the convergence in (2.8) holds along a subsequence m_k constructed in Step 3. Since F_m is just a convex combination of truncations of $\overline{F}_m, \underline{F}_m$ defined in (2.15) and (2.17), from (2.23) it follows easily that for all $(x,t) \in Q \setminus N$, $(p,X) \in \mathbb{R}^n \times S(n)$,

$$F_{m_k}(x,t,p,X) \to F(x,t,p,X) \quad \text{as} \quad k \to \infty,$$

or (2.8) holds and the proof is complete. □

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