Simulation of Fluid Flows

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Structure of the Lecture

- Physical Laws described by PDE-s'
- Discretization utilizing Finite difference method
- Two Simple Examples
  - 2D compressible potential flow about airfoil
  - Driven cavity flow utilizing Navier-Stokes equations
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Part I

Physical Laws
Physical Laws

Main Symbols

Basic Assumptions

Transport Theorem
  Control Mass and Control Volume

Governing Equations
  Conservation of Mass
  Newton’s Second Law of Motion
  The First Principle of Thermodynamics
  Differential Form
  Stress Velocity Relationship
  Volume Forces
  Equation of State
  Conservative Form of PDE

Boundary Conditions
Main Symbols

- **Pressure, density and temperature**
  
  \[ p, \quad \rho, \quad T \]

- **Velocity components**
  
  \[ \vec{V} = u\hat{i} + v\hat{j} + w\hat{k} \]

- **Normal and tangential stress**
  
  \[ \sigma, \quad \tau \]

- **Cartezian and curvilinlear coordinates**
  
  \[(x, y, z), \quad (\xi, \eta, \zeta)\]
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Basic Assumptions

- Fluid is continuous environment, thus the following assumptions have **sense**:

\[
\lim_{\Delta A \to 0} \frac{\vec{n} \cdot \Delta \vec{F}}{\Delta A} = -p \\
\lim_{\Delta A \to 0} \frac{\vec{t} \cdot \Delta \vec{F}}{\Delta A} = \tau \\
\lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} = \varrho
\]

- Thus, tools of Mathematical analysis can be **applied**!
- Fluid can be in equilibrium under arbitrary loads only when it is in **motion**.
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- Thus, tools of Mathematical analysis can be applied!
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Transport Theorem

- Control volume loose and gain various particles – physical laws are **not defined** for such case!
- Physical laws are defined for control system having **the same** particles all the time! (ControlVolume.gif)

For any global system property $B$ and property per unit mass $b = B/m$ it is valid:

$$\frac{dB}{dt} = \int_{V_{CV}} \frac{\partial}{\partial t} (\rho b) \, dV + \int_{S_{CV}} \rho b \left( \vec{V} - \vec{v} \right) \cdot \vec{n} \, dS$$

$$\frac{dB}{dt} = \int_{V_{CV}} \left\{ \frac{\partial}{\partial t} (\rho b) + \text{div} \left[ \rho b \left( \vec{V} - \vec{v} \right) \right]\right\} \, dV$$

where $\vec{v}$ is the velocity of the system boundary!
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For any global system property \( B \) and property per unit mass \( b = \frac{B}{m} \) it is valid:

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\]

where \( \vec{v} \) is the velocity of the system boundary!
At $t = 0$ control volume occupy the same space as control mass. Physical laws are derived for control mass.
Conservation of mass

\[ \frac{dm}{dt} = 0 \]

System property: \( B = m \). Property per unit mass: \( b = m/m = 1 \).

\[ \frac{dm}{dt} = \int_{\mathcal{V}_{CV}} \frac{\partial \rho}{\partial t} \, d\mathcal{V} + \oint_{S_{CV}} \rho \left( \vec{V} - \vec{v} \right) \cdot \hat{n} \, dS = 0 \]

or:

\[ \frac{dm}{dt} = \int_{\mathcal{V}_{CV}} \left\{ \frac{\partial \varrho}{\partial t} + \text{div} \left[ \varrho \left( \vec{V} - \vec{v} \right) \right] \right\} \, d\mathcal{V} = 0 \]
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Newton’s Second Law of Motion

\[
\frac{d(m \vec{V})}{dt} = \sum \vec{F} = \sum \vec{F}_S + \sum \vec{F}_V
\]

System property: \( B = m \vec{V} \). Property per unit mass:
\( b = m \vec{V}/m = \vec{V} \).

\[
\frac{d(m \vec{V})}{dt} = \int_{\nu_{CV}} \frac{\partial}{\partial t} (\rho \vec{V}) \, d\nu + \oint_{S_{CV}} \rho \vec{V} \left( \vec{V} - \vec{v} \right) \cdot \vec{n} \, dS = \sum \vec{F}_S + \sum \vec{F}_V
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where \( \sum \vec{F}_S \) are surface forces, and \( \sum \vec{F}_V \) are volume forces.
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where \( \sum \vec{F}_S \) are surface forces, and \( \sum \vec{F}_V \) are volume forces.
The First Principle of Thermodynamics

Rate of change of the energy of the system is equal to the rate of transfer of energy to the system!

\[
\frac{dE}{dt} = \frac{dQ}{dt} - \frac{dW}{dt}
\]

System property: \( B = E \). Property per unit mass: \( b = E/m = e \).

\[
\frac{dE}{dt} = \int_{V_{cv}} \frac{\partial (\rho e)}{\partial t} \ dv + \oint_{S_{cv}} \rho e \left( \vec{V} - \vec{v} \right) \cdot \vec{n} \, dS = \dot{Q} - \dot{W}
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where:

\[
e = e^o + \frac{V^2}{2} + \frac{p}{\rho} + \vec{g} \cdot \vec{r} + \cdots \quad e^o = C_V T
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Differential Form

Conservation of Mass

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0
\]

Momentum Equations

\[
\rho \frac{Du}{Dt} = \frac{\partial(-p + \tau_{xx})}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + S_x
\]

\[
\rho \frac{Dv}{Dt} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial(-p + \tau_{yy})}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + S_y
\]

\[
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\]
Differential Form

Conservation of Mass

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Differential Form

Conservation of Mass

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Momentum Equations

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Differential Form

Conservation of Mass

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0$$

Momentum Equations

$$\frac{\rho}{Dt} \frac{Du}{Dt} = \frac{\partial (-p + \tau_{xx})}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + S_x$$

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Differential Form Energy Equations

\[ \frac{\varrho}{Dt} \frac{De}{Dt} = -\nabla (\varrho \vec{V}) + \frac{\partial (u\tau_{xx})}{\partial x} + \frac{\partial (u\tau_{yx})}{\partial y} + \frac{\partial (u\tau_{zx})}{\partial z} + \frac{\partial (v\tau_{xy})}{\partial x} + \frac{\partial (v\tau_{yy})}{\partial y} + \frac{\partial (v\tau_{zy})}{\partial z} + \frac{\partial (w\tau_{xz})}{\partial x} + \frac{\partial (w\tau_{yz})}{\partial y} + \frac{\partial (w\tau_{zz})}{\partial z} + \nabla \cdot (\kappa \nabla T) + S_E \]

Mechanical Energy Equation

\[ \varrho \frac{D(V^2/2)}{Dt} = -\vec{V} \cdot \nabla p + u \left( \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) + v \left( \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right) + w \left( \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right) + \vec{V} \cdot \vec{S} \]
Differential Form Energy Equations

\[
\rho \frac{De}{Dt} = -\nabla (\rho \vec{V}) + \frac{\partial (u \tau_{xx})}{\partial x} + \frac{\partial (u \tau_{yx})}{\partial y} + \frac{\partial (u \tau_{zx})}{\partial z} + \frac{\partial (v \tau_{xy})}{\partial x} \\
+ \frac{\partial (v \tau_{yy})}{\partial y} + \frac{\partial (v \tau_{zy})}{\partial z} + \frac{\partial (w \tau_{xz})}{\partial x} + \frac{\partial (w \tau_{yz})}{\partial y} + \frac{\partial (w \tau_{zz})}{\partial z} \\
+ \nabla \cdot (\kappa \nabla T) + S_E
\]

Mechanical Energy Equation

\[
\rho \frac{D(V^2/2)}{Dt} = -\vec{V} \cdot \nabla p + u \left( \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) + \\
+ v \left( \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right) + w \left( \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right) + \vec{V} \cdot \vec{S}
\]
This assumption (valid for certain fluids) reduces number of unknowns:

\[
\begin{pmatrix}
\tau_{xx} & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \tau_{yy} & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \tau_{zz}
\end{pmatrix} = -\mu
\begin{pmatrix}
2 \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial y} & \frac{\partial u}{\partial z} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial z} \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} & 2 \frac{\partial v}{\partial y} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z}
\end{pmatrix}
\]
Volume Forces

- Gravitational Force \((S_x = 0, S_y = 0, S_z = -\rho g)\).
- Force due to electric and magnetic fields.
- Inertial Force.
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Volume Forces

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Equation of State

Under assumption of thermodynamic equilibrium we assume:

\[ p = p(\varrho, T), \quad e^o = e^o(\varrho, T) \]

for perfect gas:

\[ p = \varrho RT, \quad e^o = C_V T \]

for isentropic flow:

\[ p = p(\varrho), \quad p = p_o \left( \frac{\varrho}{\varrho_o} \right)^\kappa \]

for incompressible flow:

\[ \varrho = \text{const.} \quad \varrho = 1.225 \ [kg/m^3] \quad \text{for air} \]
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Conservative Form of PDE

Conservative form of governing equation is given as:

\[
\frac{\partial \vec{Q}}{\partial t} + \frac{\partial \vec{E}}{\partial x} + \frac{\partial \vec{F}}{\partial y} + \frac{\partial \vec{G}}{\partial z} = \frac{\partial \vec{E}_v}{\partial x} + \frac{\partial \vec{F}_v}{\partial y} + \frac{\partial \vec{G}_v}{\partial z}
\]

where:

\[
\vec{Q} = \begin{bmatrix}
\rho \\
\rho u \\
\rho v \\
\rho w \\
\rho e
\end{bmatrix}, \quad \vec{E} = \begin{bmatrix}
\rho u \\
\rho u^2 + p \\
\rho uv \\
\rho uw \\
\rho ue + pu
\end{bmatrix}, \quad \vec{F} = \begin{bmatrix}
\rho v \\
\rho uv \\
\rho v^2 + p \\
\rho vw \\
\rho ve + pv
\end{bmatrix}
\]
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\]

where:

\[
\vec{Q} = \begin{cases} 
    \varrho \\
    \varrho u \\
    \varrho v \\
    \varrho w \\
    \varrho e 
\end{cases} \quad \vec{E} = \begin{cases} 
    \varrho u \\
    \varrho u^2 + p \\
    \varrho uv \\
    \varrho uw \\
    \varrho ve + pu 
\end{cases} \quad \vec{F} = \begin{cases} 
    \varrho v \\
    \varrhouv \\
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\end{cases}
\]
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Conservative Form of PDE Continued

\[ \vec{G} = \left\{ \begin{array}{l}
\rho w \\
\rho uw \\
\rho vw \\
\rho w^2 + p \\
\rho we + pw
\end{array} \right\} \]

\[ \vec{E}_v = \left\{ \begin{array}{l}
0 \\
\tau_{xx} \\
\tau_{xy} \\
\tau_{xz} \\
\tau_{yy} \\
\tau_{yz} \\
u \tau_{xx} + v \tau_{xy} + w \tau_{xz} - q_x \\
u \tau_{yx} + v \tau_{yy} + w \tau_{yz} - q_y
\end{array} \right\} \]
Conservative Form of PDE Continued

\[ \vec{G} = \begin{cases} \rho w \\ \rho uw \\ \rho vw \\ \rho w^2 + p \\ \rho we + pw \end{cases} \]

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Conservative Form of PDE Continued

\[ \vec{G} = \begin{cases} \rho w \\ \rho u w \\ \rho v w \\ \rho w^2 + p \\ \rho w e + p w \end{cases}, \quad \vec{E}_v = \begin{cases} 0 \\ \tau_{xx} \\ \tau_{xy} \\ \tau_{xz} \\ u \tau_{xx} + v \tau_{xy} + w \tau_{xz} - q_x \end{cases} \]

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Conservative Form of PDE Continued

\[ \vec{G}_v = \begin{cases} 
0 \\
\tau_{wx} \\
\tau_{wy} \\
\tau_{wz} \\
u\tau_{zx} + v\tau_{zy} + w\tau_{zz} - q_z 
\end{cases} \]

where:

\[ \vec{q} = q_z\vec{i} + q_y\vec{j} + q_z\vec{k} = \kappa \nabla T \]
Conservative Form of PDE Continued

\[
\vec{G}_v = \begin{cases} 
0 \\
\tau_{wx} \\
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\tau_{wz} \\
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\end{cases}
\]

where:

\[
\vec{q} = q_z\hat{i} + q_y\hat{j} + q_z\hat{k} = \kappa\nabla T
\]
Typical Boundary Conditions

- **In contact with solid wall** fluid particles have the same velocity as the wall itself – **no-slip condition**.
- If the plane of symmetry exist, velocity in the plane has extreme value, while velocity normal to plane is zero – **free-slip condition**.
- **inflow condition** All components of the velocity must be known, sometimes we estimate them by solving simpler problem.
- **outflow condition** – there is no change of the velocities normal to the boundary (exit).
- **periodicity condition** – flow parameters in corresponding points must match (have the same values).
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Part II

Discretization
Discretization

Algorithm of FFS

Physical Models of Fluid Flow

Required Resources

Discretization Techniques
- FDM – Basics
- Short notation
- Explicit, Implicit, Consistency, Stability
- Convergence
Algorithm of Fluid Flow Simulations
Physical Models of Fluid Flow

FLOW MODEL

- Turbulence modeling
- Parabolized N.S.
- Thin Layer Approximation
- Boundary Layer
- Small Disturbance

FLOW SIMULATIONS

N.S.

VISCUS

INCOMPRESSIBLE

VORTEICAL

POTENTIAL

VORTEICAL

POTENTIAL

SMLAL DISTURBANCE AND LINEARIZATION

COMPRRESSIBLE

Euler

\( \nabla \times \vec{V} = 0 \)

\( \vec{V} = \text{const} \)
Required Computational Resources
Discretization Techniques

The most frequently utilized methods in engineering simulations are:

- Finite Difference Method – FDM.
- Finite Element Method – FEM.
- Finite Volume Method – FVM.

In this lecture we explain here only the most simplest method, i.e. The Finite Difference Method!
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FDM – Basics

Mathematical definition of the first derivative:

\[
\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y, z, t) - u(x, y, z, t)}{\Delta x}
\]

From the infinitely many ways to approximate the first derivative we show only three: Forward difference approximation:

\[
\frac{\partial u}{\partial x} \approx \frac{u(x + \Delta x, y, z, t) - u(x, y, z, t)}{\Delta x} + O(\Delta x)
\]

Backward difference approximation:

\[
\frac{\partial u}{\partial x} \approx \frac{u(x, y, z, t) - u(x - \Delta x, y, z, t)}{\Delta x} + O(\Delta x)
\]

Centered difference approximation:

\[
\frac{\partial u}{\partial x} \approx \frac{u(x + \Delta x, y, z, t) - u(x - \Delta x, y, z, t)}{2\Delta x} + O(\Delta x^2)
\]

forward.gif, backward.gif, centered.gif
FDM – Basics

Finite difference approximation:

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\]

forward.gif, backward.gif, centered.gif
Illustration
Since the second derivative is the first derivative of the first derivative:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{\frac{\partial u}{\partial x} (x + \Delta x/2, y, z, t) - \frac{\partial u}{\partial x} (x - \Delta x/2, y, z, t)}{\Delta x}$$

and:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(x+\Delta x, y, z, t) - u(x, y, z, t)}{\Delta x} - \frac{u(x, y, z, t) - u(x-\Delta x, y, z, t)}{\Delta x}$$

The second order derivative FD approximation is thus:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(x + \Delta x, y, z, t) - 2u(x, y, z, t) + u(x - \Delta x, y, z, t)}{(\Delta x)^2}$$
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\]
Short notation $u^n_{i,j,k} \equiv u(x_i, y_j, z_k, t_n)$

For example:

$$\frac{\partial u}{\partial t} \approx \frac{u_{i}^{n+1} - u_{i}^{n}}{\Delta t} = \frac{u(x_i, t_n + \Delta t) - u(x_i, t_n)}{\Delta t}$$

and

$$\frac{\partial u}{\partial x} \approx \frac{u_{i+1}^{n} - u_{i-1}^{n}}{2\Delta x} = \frac{u(x_i + \Delta x, t_n) - u(x_i, t_N)}{\Delta x}$$

Replacing partial derivatives in PDE we convert the problem of solving PDE to problem of solution algebraic equation!
Short notation

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Replacing partial derivatives in PDE we convert the problem of solving PDE to problem of solution algebraic equation!
Short notation $u_{i,j,k}^n \equiv u(x_i, y_j, z_k, t_n)$

For example:

$$\frac{\partial u}{\partial t} \approx \frac{u_{i}^{n+1} - u_{i}^{n}}{\Delta t} \equiv \frac{u(x_i, t_n + \Delta t) - u(x_i, t_n)}{\Delta t}$$

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Replacing partial derivatives in PDE we convert the problem of solving PDE to problem of solution algebraic equation!
Illustration of the short notation
FDM – Basics: explicit, implicit, consistency

Let’s approximate first order one-dimensional wave equation:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}$$

When derivatives are replaced with finite differences we get:

$$\frac{u_{i}^{n+1} - u_{i}^{n}}{\Delta t} = \frac{u_{i+1}^{n} - u_{i-1}^{n}}{2\Delta x}$$

or:

$$\frac{u_{i}^{n+1} - u_{i}^{n}}{\Delta t} = \frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x}$$

Left-hand side approximation is **explicit**, i.e. has only one unknown!, while right-hand side approximation is **implicit** i.e. we need to solve system of equations.

Both approximations are **consistent**!
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FDM – Basics: stability

One of possible explicit finite difference approximation of the PDE:

\[
\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}
\]

Lets rearrange both approximations into form:

\[
u^{n+1}_i = u^n_i + \frac{c}{2} \cdot (u^n_{i+1} - u^n_{i-1}), \quad c = \frac{\Delta t}{\Delta x}
\]

\[
u^{n+1}_i = u^n_i + d \cdot (u^n_{i-1} - 2u^n_i + u^n_{i+1}), \quad d = \frac{\alpha\Delta t}{(\Delta x)^2}
\]

Lets observe how error grows (dies out) for \(d = 0.49\), and \(d = 0.52\).
Animations:
dif0_49.avi, dif0_52.avi
One of possible explicit finite difference approximation of the PDE:

\[
\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{(\Delta x)^2}
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Lets rearrange both approximations into form:

\[
u_i^{n+1} = u_i^n + \frac{c}{2} \cdot (u_{i+1}^n - u_{i-1}^n), \quad c = \frac{\Delta t}{\Delta x}
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FDM – Basics: stability

One of possible explicit finite difference approximation of the PDE:

\[
\frac{u^n_{i+1} - u^n_i}{\Delta t} = \alpha \frac{u^n_{i-1} - 2u^n_i + u^n_{i+1}}{(\Delta x)^2}
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Lets rearrange both approximations into form:

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u^n_{i+1} = u^n_i + \frac{c}{2} \cdot (u^n_{i+1} - u^n_{i-1}), \quad c = \frac{\Delta t}{\Delta x}
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\[
u^n_{i+1} = u^n_i + d \cdot (u^n_{i-1} - 2u^n_i + u^n_{i+1}), \quad d = \frac{\alpha \Delta t}{(\Delta x)^2}
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FDM – Basics: convergence

Finite difference scheme converges (according to Lax) if:

- Approximation is consistent!
- Approximation is stable!
FDM – Basics: convergence

Finite difference scheme converges (according to Lax) if:

▶ **Approximation is consistent!**

▶ **Approximation is stable!**
FDM – Basics: convergence

Finite difference scheme converges (according to Lax) if:

- Approximation is consistent!
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Part III

Examples
Examples

Example 1
Problem Statement
Boundary Conditions
Solution Procedure
Grid Generation
Metrics of transformation
Approximation of the PDE
Solution procedure
Typical Result

Example 2
Example 2, Problem Statement
Boundary Conditions
Approximation of PDEs
Approximation of BCs
Computational Procedure
Results
Physical Models of Fluid Flow

FLOW MODEL

Turbulence modeling
Parabolized N.S.
Thin Layer Approximation
Boundary Layer
Small Disturbance

VERY SLOW

INCOMPRESSIBLE

VISCOUS

COMPRESSIBLE

FLOW SIMULATIONS

N.S.

\( \nu = 0 \)

INVISCID

\( \rho = \text{const} \)

Euler

\( \nabla \times \vec{V} = 0 \)

INCOMPRESSIBLE

VORTICAL

\nabla \times \vec{V} = 0

POTENTIAL

COMPRESSIBLE

VORTICAL

\nabla \times \vec{V} = 0

POTENTIAL

SMALL DISTURBANCE AND LINEARIZATION
Potential compressible two-dimensional flow is described by conservation of mass equation:

$$\frac{\partial}{\partial x} \left( \rho \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left( \rho \frac{\partial \phi}{\partial y} \right) = 0$$

and relationship between speed and density:

$$\frac{\rho_o}{\rho} = \left( 1 + \frac{\gamma - 1}{2} M^2 \right)^{1/(\gamma - 1)}$$

where

$$\tilde{V} = u \tilde{i} + v \tilde{j}, \quad V = \sqrt{u^2 + v^2}$$

and:

$$u = \frac{\partial \phi}{\partial x} \equiv \phi_x, \quad v = \frac{\partial \phi}{\partial y} \equiv \phi_y$$
Example 1 – Problem statement 1/2

Potential compressible two-dimensional flow is described by conservation of mass equation:

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\frac{\partial}{\partial x} \left( \rho \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left( \rho \frac{\partial \phi}{\partial y} \right) = 0
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and relationship between speed and density:

\[
\frac{\rho_o}{\rho} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{1/(\gamma-1)}, \quad M = \frac{V}{a}
\]

where

\[
\vec{V} = u \vec{i} + v \vec{j}, \quad V = \sqrt{u^2 + v^2}
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and:

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and relationship between speed and density:

$$\frac{\rho_0}{\rho} = \left( 1 + \frac{\gamma - 1}{2} M^2 \right)^{1/(\gamma - 1)}$$

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\]

and:

\[
u = \frac{\partial \phi}{\partial y} \equiv \phi_y
\]
Example 1 – Problem statement 2/2

Non-dimensionalizing with critical speed of sound doesn’t change the first equation:

\[
\frac{\partial}{\partial x} \left( \rho \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left( \rho \frac{\partial \phi}{\partial y} \right) = 0
\]

Here we used short notation for derivatives:

\[
\phi_x \equiv \frac{\partial \phi}{\partial x}, \quad \phi_y \equiv \frac{\partial \phi}{\partial y}, \quad \phi_{xx} \equiv \frac{\partial^2 \phi}{\partial x^2}
\]

Nondimensional velocities:

\[
\frac{u_\infty}{a^*} = M \sqrt{\frac{\gamma + 1}{2 + (\gamma - 1) M^2}} \cos \alpha \quad \frac{v_\infty}{a^*} = M \sqrt{\frac{\gamma + 1}{2 + (\gamma - 1) M^2}} \sin \alpha
\]

Density equation reduces to:

\[
\frac{\rho}{\rho_o} = \left[ 1 - \frac{\gamma - 1}{\gamma + 1} \cdot (u^{*2} + v^{*2}) \right]^{1/(\gamma - 1)}
\]
Example 1 – Problem statement 2/2

Non-dimensionalizing with critical speed of sound doesn’t change the first equation:

\[(\varrho \phi_x)_x + (\varrho \phi_y)_y = 0\]

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Non-dimensionalizing with critical speed of sound doesn’t change the first equation:

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$$\frac{\rho}{\rho_\infty} = \left[ 1 - \frac{\gamma - 1}{\gamma + 1} \cdot (u^{*2} + v^{*2}) \right]^{1/(\gamma - 1)}$$
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Non-dimensionalizing with critical speed of sound doesn’t change the first equation:

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\[\frac{u_\infty}{a^*} = M \sqrt{\frac{\gamma + 1}{2 + (\gamma - 1)M^2}} \cos \alpha \quad \quad \quad \frac{v_\infty}{a^*} = M \sqrt{\frac{\gamma + 1}{2 + (\gamma - 1)M^2}} \sin \alpha\]

Density equation reduces to:

\[\frac{\varrho}{\varrho_0} = \left[1 - \frac{\gamma - 1}{\gamma + 1} \cdot (u^{*2} + v^{*2})\right]^{1/(\gamma - 1)}\]
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Non-dimensionalizing with critical speed of sound doesn’t change the first equation:

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Density equation reduces to:

\[\frac{\varrho}{\varrho_o} = \left[ 1 - \frac{\gamma - 1}{\gamma + 1} \cdot (u^*^2 + v^*^2) \right]^{1/(\gamma - 1)}\]
Boundary Conditions

\[ \phi = U_\infty x + V_\infty y + \frac{\Gamma}{2\pi} \arctg \frac{y}{x} \]

\[ \frac{\partial \phi}{\partial n} = 0 \]

\[ (\rho \phi)_x + (\rho \phi)_y = 0 \]
Question of uniqueness

Any combination of these two flows satisfy all boundary conditions!
Uniqueness is achieved by imposing Kutta-Joukowsky condition.
Question of uniqueness

Any combination of these two flows satisfy all boundary conditions! Uniqueness is achieved by imposing Kutta-Joukowsky conditions.
Solution Procedure 1/2

Solution has the jump of the potential what is not desirable property, we can split the solution in two parts: one which will correctly describe jump and other which is continuous and supplements the first part to the complete solution!

\[ \phi = \varphi + \frac{\Gamma}{2\pi} \arctan \frac{y}{x} = \varphi + \Gamma \cdot \Phi \]
Solution Procedure 1/2

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\[ \phi = \varphi + \frac{\Gamma}{2\pi} \arctan \frac{y}{x} = \varphi + \Gamma \cdot \Phi \]
Solution Procedure 2/2

Our governing equation now looks:

\[ (\varrho \varphi_x)_x + (\varrho \varphi_y)_y = -(\varrho \Phi_x)_x + (\varrho \Phi_y)_y - \Gamma \cdot \varrho \cdot f(x, y) \]

where \( f(x, y) \) is known over the physical region!

Boundary conditions:

\[
\frac{\partial \varphi}{\partial n} = \frac{\partial \varphi + \Gamma \cdot \Phi}{\partial n} = 0, \quad \Rightarrow \quad \frac{\partial \varphi}{\partial n} = -\Gamma \cdot \frac{\partial \Phi}{\partial n}
\]

\[ \varphi = u_\infty \cdot x + v_\infty \cdot y, \quad \Phi = \frac{1}{2\pi} \arctan \frac{y}{x} \]

\( \varphi \) is continuous over cut!
Solution Procedure 2/2

Our governing equation now looks:

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(\varrho \varphi_x)_x + (\varrho \varphi_y)_y = - (\varrho \Phi_x)_x + (\varrho \Phi_y)_y - \Gamma \cdot \varrho \cdot f(x, y)
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Our governing equation now looks:

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\]

where \( f(x,y) \) is known over the physical region!

Boundary conditions:

\[
\frac{\partial \varphi}{\partial n} = \varphi' + \Gamma \cdot \Phi \frac{\partial \Phi}{\partial n} = 0, \quad \Rightarrow \quad \frac{\partial \varphi}{\partial n} = -\Gamma \cdot \frac{\partial \Phi}{\partial n}
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\varphi = u_\infty \cdot x + v_\infty \cdot y, \quad \Phi = \frac{1}{2\pi} \arctan \frac{y}{x}
\]

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Solution Procedure 2/2

Our governing equation now looks:

$$\left( \varrho \varphi_x \right)_x + \left( \varrho \varphi_y \right)_y = -\left( \varrho \Phi_x \right)_x + \left( \varrho \Phi_y \right)_y - \Gamma \cdot \varrho \cdot f(x,y)$$

where $f(x, y)$ is known over the physical region!

Boundary conditions:

$$\frac{\partial \varphi}{\partial n} = \frac{\partial \varphi + \Gamma \cdot \Phi}{\partial n} = 0, \quad \Rightarrow \quad \frac{\partial \varphi}{\partial n} = -\Gamma \cdot \frac{\partial \Phi}{\partial n}$$

$$\varphi = u_\infty \cdot x + v_\infty \cdot y, \quad \Phi = \frac{1}{2\pi} \arctan \frac{y}{x}$$

$\varphi$ is continuous over cut!
Our governing equation now looks:

\[
\left(:\rho \varphi_x\right)_x + \left(:\rho \varphi_y\right)_y = -\left(:\rho \Phi_x\right)_x + \left(:\rho \Phi_y\right)_y - \Gamma \cdot \rho \cdot f(x, y)
\]

where \(f(x, y)\) is known over the physical region!

Boundary conditions:

\[
\frac{\partial \varphi}{\partial n} = \frac{\partial \varphi}{\partial n} + \Gamma \cdot \Phi = 0, \quad \Rightarrow \quad \frac{\partial \varphi}{\partial n} = -\Gamma \cdot \frac{\partial \Phi}{\partial n}
\]

\[
\varphi = u_\infty \cdot x + v_\infty \cdot y, \quad \Phi = \frac{1}{2\pi} \arctan \frac{y}{x}
\]

\(\varphi\) is continuous over cut!
Grid Generation

\[ \left( \frac{\rho U}{j} \right)_x + \left( \frac{\rho V}{j} \right)_y = 0 \]

Boundary only in computational space (CUT)

Zero flow through airfoil
Closer look at physical space
Grid generation by solving PDEs

We describe the grid as relationship of the form

\[ \xi = \xi(x, y), \quad \eta = \eta(x, y) \]

Grids are generated by solving:

\[
\nabla^2 \xi = 0 \quad \Rightarrow \quad a\xi_{\xi\xi} + 2b\xi_{\xi\eta} + c\eta_{\eta\eta} = 0 \\
\nabla^2 \eta = 0 \quad \Rightarrow \quad a\eta_{\xi\xi} + 2b\eta_{\xi\eta} + c\eta_{\eta\eta} = 0
\]

\[
a = J^2(x_\eta^2 + y_\eta^2) \\
b = -J^2(x_\xi x_\eta + y_\xi y_\eta) \\
c = J^2(x_\xi^2 + y_\xi^2) \\
J = \frac{1}{x_\xi y_\eta - x_\eta y_\xi}
\]

We need the guess for the coefficients \(a, b,\) and \(c\) (algebraic grid).
Grid generation by solving PDEs

We describe the grid as relationship of the form

\[ \xi = \xi(x, y), \quad \eta = \eta(x, y) \]

Grids are generated by solving:

\[ \nabla^2 \xi = 0 \quad \Rightarrow \quad ax_{\xi \xi} + 2bx_{\xi \eta} + cx_{\eta \eta} = 0 \]
\[ \nabla^2 \eta = 0 \quad \Rightarrow \quad ay_{\xi \xi} + 2by_{\xi \eta} + cy_{\eta \eta} = 0 \]

\[ a = J^2(x_\eta^2 + y_\eta^2) \]
\[ b = -J^2(x_\xi x_\eta + y_\xi y_\eta) \]
\[ c = J^2(x_\xi^2 + y_\xi^2) \]
\[ J = \frac{1}{x_\xi y_\eta - x_\eta y_\xi} \]

We need the guess for the coefficients \( a, b, \) and \( c \) (algebraic grid).
Grid generation by solving PDEs

We describe the grid as relationship of the form

\[ \xi = \xi(x, y), \quad \eta = \eta(x, y) \]

Grids are generated by solving:

\[
\begin{align*}
\nabla^2 \xi &= 0 \implies ax_{\xi\xi} + 2bx_{\xi\eta} + cx_{\eta\eta} = 0 \\
\nabla^2 \eta &= 0 \implies ay_{\xi\xi} + 2by_{\xi\eta} + cy_{\eta\eta} = 0
\end{align*}
\]

\[
\begin{align*}
a &= J^2(x_\eta^2 + y_\eta^2) \\
b &= -J^2(x_\xi x_\eta + y_\xi y_\eta) \\
c &= J^2(x_\xi^2 + y_\xi^2) \\
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\end{align*}
\]

We need the guess for the coefficients \( a, b, \) and \( c \) (algebraic grid)
Metrics of transformation

In the computational space our PDE looks:

\[
\left( \frac{\varrho U}{J} \right)_\xi + \left( \frac{\varrho V}{J} \right)_\eta = 0
\]

together with

\[
\varrho = \left[ 1 - \frac{\gamma - 1}{\gamma + 1} (U\phi_\xi + V\phi_\eta) \right]^{1/(\gamma - 1)}
\]

On the airfoil:

\[
\left. \frac{\partial \phi}{\partial n} \right|_{\eta = \text{const}} = \frac{b\phi_\xi + c\phi_\eta}{\sqrt{c}} = 0
\]

So we need \( J, a, b, \) and \( c \) to approximate our PDE in the computational space!
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So we need \( J, a, b, \) and \( c \) to approximate our PDE in the computational space!
How these metrics look 1/4
How these metrics look 2/4
How these metrics look 3/4
How these metrics look 4/4
Approximation of the PDE 1/3

Our PDE at the point \((i, j)\) in the computational space looks:

\[
\left(\frac{\varrho U}{J}\right)_{\xi_{i,j}} + \left(\frac{\varrho V}{J}\right)_{\eta_{i,j}} = 0
\]

where:

\[
U = a\phi_\xi + b\phi_\eta, \quad V = b\phi_\xi + c\phi_\eta
\]

Since \(\phi = \varphi + \Gamma \cdot \Phi\):

\[
\left(\frac{\varrho U}{J}\right)_{\xi_{i,j}} + \left(\frac{\varrho V}{J}\right)_{\eta_{i,j}} = -\left(\frac{\varrho \tilde{U}}{J}\right)_{\xi_{i,j}} - \left(\frac{\varrho \tilde{V}}{J}\right)_{\eta_{i,j}} = \Gamma \cdot \varrho_{i,j} \cdot f(\xi, \eta)_{i,j}
\]

where now:

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Approximation of the PDE 1/3

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\]

where now:

\[
U = a\varphi\xi + b\varphi\eta, \quad V = b\varphi\xi + c\varphi\eta
\]
Approximation of the PDE 2/3

Approximating the former equation with the finite differences we have:

\[
\left( \frac{\partial U}{J} \right)_{i+1/2,j} - \left( \frac{\partial U}{J} \right)_{i-1/2,j} + \left( \frac{\partial V}{J} \right)_{i,j+1/2} - \left( \frac{\partial V}{J} \right)_{i,j-1/2} = \Gamma \cdot \varrho_{i,j} \cdot f_{i,j}
\]

where:

\[
U_{i+1/2,j} \approx a_{i+1/2,j} (\varphi_{i+1,j} - \varphi_{i,j}) + b_{i+1/2,j} \frac{\varphi_{i+1,j+1} + \varphi_{i,j+1} - \varphi_{i+1,j-1} - \varphi_{i,j-1}}{4}
\]

\[
U_{i-1/2,j} \approx a_{i-1/2,j} (\varphi_{i,j} - \varphi_{i-1,j}) + b_{i-1/2,j} \frac{\varphi_{i-1,j+1} + \varphi_{i,j+1} - \varphi_{i-1,j-1} - \varphi_{i,j-1}}{4}
\]
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\[
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\]

where:

\[
U_{i+1/2,j} \approx a_{i+1/2,j}(\varphi_{i+1,j} - \varphi_{i,j}) + \frac{b_{i+1/2,j}}{4} \frac{\varphi_{i+1,j+1} + \varphi_{i,j+1} - \varphi_{i+1,j-1} - \varphi_{i,j-1}}{4} \\
U_{i-1/2,j} \approx a_{i-1/2,j}(\varphi_{i,j} - \varphi_{i-1,j}) + \frac{b_{i-1/2,j}}{4} \frac{\varphi_{i-1,j+1} + \varphi_{i,j+1} - \varphi_{i-1,j-1} - \varphi_{i,j-1}}{4}
\]
Approximation of the PDE 3/3

and also:

\[
V_{i,j+1/2} \approx b_{i,j+1/2} \frac{\varphi_{i+1,j+1} + \varphi_{i+1,j} - \varphi_{i-1,j+1} - \varphi_{i-1,j}}{4} + \\
+ c_{i,j+1/2} \cdot (\varphi_{i,j+1} - \varphi_{i,j})
\]

\[
V_{i,j-1/2} \approx b_{i,j-1/2} \frac{\varphi_{i+1,j-1} + \varphi_{i+1,j} - \varphi_{i-1,j-1} - \varphi_{i-1,j}}{4} + \\
+ c_{i,j-1/2} \cdot (\varphi_{i,j} - \varphi_{i,j-1})
\]

After substituting all of this approximations we get the following:

\[
\mathcal{A} \varphi_{i,j}^{k+1} = \mathcal{L}(\varphi^k) \implies \varphi_{i,j}^{k+1} = \frac{1}{\mathcal{A}} \mathcal{L}(\varphi^k)
\]

where:

\[
\mathcal{A} = \left( \frac{\varrho}{j} a \right)_{i+1/2,j} + \left( \frac{\varrho}{j} a \right)_{i-1/2,j} + \left( \frac{\varrho}{j} c \right)_{i,j+1/2} + \left( \frac{\varrho}{j} c \right)_{i,j-1/2}
\]
Approximation of the PDE 3/3

and also:

\[ V_{i,j+1/2} \approx b_{i,j+1/2} \frac{\varphi_{i+1,j+1} + \varphi_{i+1,j} - \varphi_{i-1,j+1} - \varphi_{i-1,j}}{4} + \]
\[ + c_{i,j+1/2} \cdot (\varphi_{i,j+1} - \varphi_{i,j}) \]

\[ V_{i,j-1/2} \approx b_{i,j-1/2} \frac{\varphi_{i+1,j-1} + \varphi_{i+1,j} - \varphi_{i-1,j-1} - \varphi_{i-1,j}}{4} + \]
\[ + c_{i,j-1/2} \cdot (\varphi_{i,j} - \varphi_{i,j-1}) \]

After substituting all of this approximations we get the following:

\[ A\varphi_{i,j}^{k+1} = \mathcal{L}(\varphi^k) \quad \Rightarrow \quad \varphi_{i,j}^{k+1} = \frac{1}{A} \mathcal{L}(\varphi^k) \]

where:

\[ A = \left( \frac{\rho}{j} a \right)_{i+1/2,j} + \left( \frac{\rho}{j} a \right)_{i-1/2,j} + \left( \frac{\rho}{j} c \right)_{i,j+1/2} + \left( \frac{\rho}{j} c \right)_{i,j-1/2} \]
Approximation of the PDE 3/3

and also:

\[
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\]
\[
+ c_{i,j+1/2} \cdot (\varphi_{i,j+1} - \varphi_{i,j}) 
\]

\[
V_{i,j-1/2} \approx b_{i,j-1/2} \frac{\varphi_{i+1,j-1} + \varphi_{i+1,j} - \varphi_{i-1,j-1} - \varphi_{i-1,j}}{4} + 
\]
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After substituting all of this approximations we get the following:

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A\varphi_{i,j}^{k+1} = \mathcal{L}(\varphi^k) \quad \Rightarrow \quad \varphi_{i,j}^{k+1} = \frac{1}{A} \mathcal{L}(\varphi^k) 
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\]
Solution procedure 1/3

Subtract from both side of the equation:

$$\varphi_{i,j}^{k+1} = \frac{1}{A} \mathcal{L}(\varphi^k)$$

We get the correction from iteration $k$ to the iteration $k+1$

$$\Delta \varphi_{i,j}^k = \varphi_{i,j}^{k+1} - \varphi_{i,j}^k = -\varphi_{i,j}^k + \frac{\mathcal{L}}{A}(\varphi^k)$$

Let us now magnify the correction $\omega$ times:

$$\varphi_{i,j}^{k+1} = \varphi_{i,j}^k + \omega \Delta \varphi_{i,j}^k = (1 - \omega)\varphi_{i,j}^k + \frac{\omega}{A} \mathcal{L}(\varphi^k)$$

We got so called SOR procedure
Solution procedure 1/3

Subtract from both side of the equation:

$$\varphi_{i,j}^{k+1} = \frac{1}{A} \mathcal{L}(\varphi^k)$$

$\varphi_{i,j}^k$, we will get the correction from iteration $k$ to the iteration $k + 1$

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\[ \varphi_{i,j}^{k+1} = \frac{1}{\mathcal{A}} \mathcal{L}(\varphi^k) \]

\( \varphi_{i,j}^k \), we will get the correction from iteration \( k \) to the iteration \( k + 1 \)

\[ \Delta \varphi_{i,j}^k = \varphi_{i,j}^{k+1} - \varphi_{i,j}^k = -\varphi_{i,j}^k + \frac{\mathcal{L}}{\mathcal{A}}(\varphi^k) \]

let us now magnify the correction \( \omega \) times:

\[ \varphi_{i,j}^{k+1} = \varphi_{i,j}^k + \omega \Delta \varphi_{i,j}^k = (1 - \omega)\varphi_{i,j}^k + \frac{\omega}{\mathcal{A}} \mathcal{L}(\varphi^k) \]

we got so called SOR procedure
Solution procedure 2/3

At the coordinate line $j = 1$ we do not have values corresponding to the $j - 1$, so

- On the cut use the fact that the points on the other side of the computational space are below (above) the current point
- On the airfoil we need to use one sided approximation:

$$
\frac{\partial(\varphi + \Gamma \cdot \Phi)}{\partial n} = 0, \quad \Rightarrow \quad \frac{\partial \varphi}{\partial n} = -\Gamma \cdot \frac{\partial \Phi}{\partial n} = \Gamma \cdot \gamma(x, y)
$$

where

$$
\Phi = \frac{1}{2\pi} \arctan \frac{y}{x}
$$

Circulation is determined from

$$
\Gamma^{k+1} = \Gamma^k + \delta(V_\star - V^\star)
$$
Solution procedure 2/3

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where

$$\Phi = \frac{1}{2\pi} \arctan \frac{y}{x}$$

Circulation is determined from

$$\Gamma^{k+1} = \Gamma^k + \delta(V^* - V^*)$$
Solution procedure 2/3

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$$\frac{\partial (\varphi + \Gamma \cdot \Phi)}{\partial n} = 0, \quad \Rightarrow \quad \frac{\partial \varphi}{\partial n} = -\Gamma \cdot \frac{\partial \Phi}{\partial n} = \Gamma \cdot \gamma(x, y)$$

where

$$\Phi = \frac{1}{2\pi} \arctan \frac{y}{x}$$

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Solution procedure 2/3

At the coordinate line $j = 1$ we do not have values corresponding to the $j - 1$, so

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- On the airfoil we need to use one sided approximation:

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\frac{\partial (\varphi + \Gamma \cdot \Phi)}{\partial n} = 0, \quad \Rightarrow \quad \frac{\partial \varphi}{\partial n} = -\Gamma \cdot \frac{\partial \Phi}{\partial n} = \Gamma \cdot \gamma(x, y)
$$

where

$$
\Phi = \frac{1}{2\pi} \arctan \frac{y}{x}
$$

Circulation is determined from

$$
\Gamma^{k+1} = \Gamma^k + \delta(V^*_k - V^*)
$$
Solution procedure 3/3

In the computational space normal derivative looks:

$$\frac{\partial \varphi}{\partial n} = \frac{1}{\sqrt{c}} \left( b \varphi_\xi + c \varphi_\eta \right) = \Gamma \cdot \gamma(x, y)$$

On the airfoil

$$b_{i,1} \frac{\varphi_{i+1,1} - \varphi_{i-1,1}}{2} + c_{i,1} \frac{-3 \varphi_{i,1} + 4 \varphi_{i,2} - \varphi_{i,3}}{4} = \Gamma \cdot \sqrt{c_{i,j}} \cdot \gamma(x_{i,1}, y_{i,1})$$

From the equation above we can express $\varphi_{i,1}$ in order to obtain new guess for the value of the velocity potential at the airfoil surface!
In the computational space normal derivative looks:

\[
\frac{\partial \varphi}{\partial n} = \frac{1}{\sqrt{c}} (b \varphi_\xi + c \varphi_\eta) = \Gamma \cdot \gamma(x, y)
\]

On the airfoil

\[
b_{i,1} \frac{\varphi_{i+1,1} - \varphi_{i-1,1}}{2} + c_{i,1} \frac{-3\varphi_{i,1} + 4\varphi_{i,2} - \varphi_{i,3}}{4} = \Gamma \cdot \sqrt{c_{i,j}} \cdot \gamma(x_{i,1}, y_{i,1})
\]

From the equation above we can express \( \varphi_{i,1} \) in order to obtain new guess for the value of the velocity potential at the airfoil surface!
In the computational space normal derivative looks:

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\frac{\partial \varphi}{\partial n} = \frac{1}{\sqrt{c}} (b \varphi_\xi + c \varphi_\eta) = \Gamma \cdot \gamma(x, y)
\]

On the airfoil

\[
b_{i,1} \frac{\varphi_{i+1,1} - \varphi_{i-1,1}}{2} + c_{i,1} \frac{-3 \varphi_{i,1} + 4 \varphi_{i,2} - \varphi_{i,3}}{4} = \Gamma \cdot \sqrt{c_{i,j}} \cdot \gamma(x_{i,1}, y_{i,1})
\]

From the equation above we can express \(\varphi_{i,1}\) in order to obtain new guess for the value of the velocity potential at the airfoil surface!
Typical Result

NACA 4415
alpha=4
Example 2, Problem Statement

Nondimensional form of 2D N-S Equations look:

\[
\frac{\partial \Omega}{\partial t} + u \frac{\partial \Omega}{\partial x} + v \frac{\partial \Omega}{\partial y} = \frac{1}{Re_\infty} \left( \frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2} \right)
\]

\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\Omega, \quad u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}
\]

\(u, v\) are \(x\)- and \(y\)-component of the velocity; \(\Omega = \nabla \times \vec{V}\) is vorticity, \(\psi\) - stream function, \(t\) - time, and

\[
Re_\infty = \frac{u_o L}{\nu}, \quad \Omega = \frac{\Omega L}{u_o}, \quad \psi = \frac{\psi}{u_o L}
\]

\[
t = t \cdot \frac{u_o}{L}, \quad x = \frac{x}{L}, \quad y = \frac{y}{L}
\]
Boundary Conditions

Along boundary $ABCD$

$\psi =$ const., and derivatives of the $\psi$ along boundary are zero.

Along $\overline{AB}$:

$$\psi = 0$$

$$\Omega_{i,1} = -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = -\frac{\partial^2 \psi}{\partial y^2}$$

$$u = \frac{\partial \psi}{\partial y} = 0, \quad v = -\frac{\partial \psi}{\partial x} = 0$$

Along $\overline{BC}$

$$\psi = 0$$

$$\Omega(l,j) = -\frac{\partial^2 \psi}{\partial x^2}$$

$$u = \frac{\partial \psi}{\partial y} = 0, \quad v = -\frac{\partial \psi}{\partial x}$$
Boundary Conditions

Along boundary $ABCD$

$\psi =$ const., and derivatives of the $\psi$ along boundary are zero.

Along $\overline{AB}$:

\[
\psi = 0
\]

\[
\Omega_{i,1} = -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = -\frac{\partial^2 \psi}{\partial y^2}
\]

\[
u = \frac{\partial \psi}{\partial y} = 0, \quad v = -\frac{\partial \psi}{\partial x} = 0
\]

Along $\overline{BC}$

$\psi = 0$

\[
\Omega(i,j) = -\frac{\partial^2 \psi}{\partial x^2}
\]

\[
u = \frac{\partial \psi}{\partial y} = 0, \quad v = -\frac{\partial \psi}{\partial x} = 0
\]
Boundary Conditions

Along boundary $ABCD$

$\psi = \text{const.}$, and derivatives of the $\psi$ along boundary are zero.

Along $\overline{AB}$:

$\psi = 0$

$\Omega_{i,1} = -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = -\frac{\partial^2 \psi}{\partial y^2}$

$u = \frac{\partial \psi}{\partial y} = 0, \quad v = -\frac{\partial \psi}{\partial x} = 0$

Along $\overline{BC}$

$\psi = 0$

$\Omega(l,j) = -\frac{\partial^2 \psi}{\partial x^2}$

$u = \frac{\partial \psi}{\partial y} = 0, \quad v = -\frac{\partial \psi}{\partial x} = 0$
Boundary Conditions Continued

Along \( \overline{CD} \)

\[
\psi = 0 \\
\Omega_{i,j} = -\frac{\partial^2 \psi}{\partial y^2} \\
\frac{\partial \psi}{\partial y} = u_0, \quad v = -\frac{\partial \psi}{\partial x} = 0
\]

Along \( \overline{DA} \)

\[
\psi = 0 \\
\Omega(1,j) = -\frac{\partial^2 \psi}{\partial x^2} \\
\frac{\partial \psi}{\partial y} = 0, \quad v = -\frac{\partial \psi}{\partial x} = 0
\]
Boundary Conditions Continued

Along $\overline{CD}$

$$\psi = 0$$

$$\Omega_{i,j} = -\frac{\partial^2 \psi}{\partial y^2}$$

$$u = \frac{\partial \psi}{\partial y} = u_o, \quad v = -\frac{\partial \psi}{\partial x} = 0$$

Along $\overline{DA}$

$$\psi = 0$$

$$\Omega(1,j) = -\frac{\partial^2 \psi}{\partial x^2}$$

$$u = \frac{\partial \psi}{\partial y} = 0, \quad v = -\frac{\partial \psi}{\partial x} = 0$$
Approximation of PDEs

\[
\begin{align*}
\frac{\Omega_{i,j}^{n+1} - \Omega_{i,j}^n}{\Delta t} + u_{i,j}^n \frac{\Omega_{i+1,j}^n - \Omega_{i-1,j}^n}{2\Delta x} + v_{i,j}^n \frac{\Omega_{i,j+1}^n - \Omega_{i,j-1}^n}{2\Delta y} &= \\
= \frac{1}{Re_\infty} \left[ \frac{\Omega_{i+1,j}^n - 2\Omega_{i,j}^n + \Omega_{i-1,j}^n}{(\Delta x)^2} + \frac{\Omega_{i,j+1}^n - 2\Omega_{i,j}^n + \Omega_{i,j-1}^n}{(\Delta y)^2} \right] \\
\end{align*}
\]

\[
\bar{\psi}_{i,j}^{k+1} = \frac{(\Delta x)^2 \Omega_{i,j}^k + \psi_{i+1,j}^k + \psi_{i-1,j}^k + \beta^2 \left( \psi_{i,j+1}^k + \psi_{i,j-1}^k \right)}{2(1 + \beta^2)}
\]

\[
\psi_{i,j}^{k+1} = \omega \psi_{i,j}^k + (1 - \omega) \bar{\psi}_{i,j}^{k+1}
\]

\[
\beta = \frac{\Delta x}{\Delta y}, \quad u_{i,j} = \frac{\psi_{i,j+1} - \psi_{i,j-1}}{2\Delta y}, \quad v_{i,j} = -\frac{\psi_{i+1,j} - \psi_{i-1,j}}{2\Delta x}
\]
Approximation of PDEs

\[
\begin{align*}
\frac{\Omega_{i,j}^{n+1} - \Omega_{i,j}^n}{\Delta t} + u_{i,j}^n \frac{\Omega_{i+1,j}^n - \Omega_{i-1,j}^n}{2\Delta x} + v_{i,j}^n \frac{\Omega_{i,j+1}^n - \Omega_{i,j-1}^n}{2\Delta y} &= \\
= \frac{1}{Re_\infty} \left[ \frac{\Omega_{i+1,j}^n - 2\Omega_{i,j}^n + \Omega_{i-1,j}^n}{(\Delta x)^2} + \frac{\Omega_{i,j+1}^n - 2\Omega_{i,j}^n + \Omega_{i,j-1}^n}{(\Delta y)^2} \right]
\end{align*}
\]

\[
\bar{\psi}_{i,j}^{k+1} = \frac{(\Delta x)^2 \Omega_{i,j}^k + \psi_{i+1,j}^k + \psi_{i-1,j}^k + \beta^2 (\psi_{i,j+1}^k + \psi_{i,j-1}^k)}{2(1 + \beta^2)}
\]

\[
\psi_{i,j}^{k+1} = \omega \psi_{i,j}^k + (1 - \omega) \bar{\psi}_{i,j}^{k+1}
\]

\[
\beta = \frac{\Delta x}{\Delta y}, \quad u_{i,j} = \frac{\psi_{i,j+1} - \psi_{i,j-1}}{2\Delta y}, \quad v_{i,j} = -\frac{\psi_{i+1,j} - \psi_{i-1,j}}{2\Delta x}
\]
Approximation of BCs

Along $\overline{AB}$

$$\psi_{i,1} = 0$$

Taylor expansion gives

$$\psi_{i,2} = \psi_{i,1} + \left. \frac{\partial \psi}{\partial y} \right|_{i,1} \Delta y + \left. \frac{\partial^2 \psi}{\partial y^2} \right|_{i,1} \frac{(\Delta y)^2}{2} + \ldots$$

$$\Omega_{i,1} = 2 \cdot \frac{\psi_{i,1} - \psi_{i,2}}{(\Delta y)^2} = -2 \cdot \frac{\psi_{i,2}}{(\Delta y)^2}$$

Similarly along $\overline{BC}$ and $\overline{AD}$!
Approximation of BCs

Along $AB$

$\psi_{i,1} = 0$

Taylor expansion gives

$$\psi_{i,2} = \psi_{i,1} + \frac{\partial \psi}{\partial y}_{i,1} \Delta y + \left. \frac{\partial^2 \psi}{\partial y^2} \right|_{i,1} \frac{(\Delta y)^2}{2} + \ldots$$

$$\Omega_{i,1} = 2 \cdot \frac{\psi_{i,1} - \psi_{i,2}}{(\Delta y)^2} = -2 \cdot \frac{\psi_{i,2}}{(\Delta y)^2}$$

Similarly along $BC$ and $AD$!
Approximation of BCs Continued

Along $\overline{BC}$

\[ \psi(I, j) = 0 \]
\[ \Omega_{I,j} = -2 \cdot \frac{\psi_{I-1,j}}{(\Delta x)^2} \]

Along $\overline{AD}$

\[ \psi(1, j) = 0 \]
\[ \Omega_{1,j} = -2 \cdot \frac{\psi_{2,j}}{(\Delta x)^2} \]
Approximation of BCs Continued

Along \( \overline{BC} \)

\[ \psi(I, j) = 0 \]
\[ \Omega_{I,j} = -2 \cdot \frac{\psi_{I-1,j}}{(\Delta x)^2} \]

Along \( \overline{AD} \)

\[ \psi(1, j) = 0 \]
\[ \Omega_{1,j} = -2 \cdot \frac{\psi_{2,j}}{(\Delta x)^2} \]
Approximation of BCs Continued

Along $\overline{CD}$ we have to take into account that

$$u = \frac{\partial \psi}{\partial y} = u_o$$

Taylor expansion:

$$\psi_{J-1,i} = \psi_{J,i} + \frac{\partial \psi}{\partial y} \bigg|_{J,i} (-\Delta y) + \frac{\partial^2 \psi}{\partial y^2} \bigg|_{J,i} \frac{(\Delta y)^2}{2}$$

from where:

$$\Omega_{J,i} = -2 \frac{\psi_{J-1,i}}{(\Delta y)^2} - 2 \cdot \frac{u_o}{\Delta y}$$
Approximation of BCs Continued

Along $\overline{CD}$ we have to take into account that

$$u = \frac{\partial \psi}{\partial y} = u_o$$

Taylor expansion:

$$\psi_{J-1,i} = \psi_{J,i} + \left( \frac{\partial \psi}{\partial y} \right)_{J,i}^{|y=J,i} (-\Delta y) + \left( \frac{\partial^2 \psi}{\partial y^2} \right)_{J,i}^{|y=J,i} \frac{(\Delta y)^2}{2}$$

from where:

$$\Omega_{J,i} = -2 \frac{\psi_{J-1,i}}{(\Delta y)^2} - 2 \cdot \frac{u_o}{\Delta y}$$
Computational Procedure

1. Initialize all variables corresponding to $t = 0$

2. Calculate velocities inside the cavity

\[
\begin{align*}
u_{i,j} &= \frac{\psi_{i,j+1} - \psi_{i,j-1}}{2\Delta y} \\
v_{i,j} &= \frac{\psi_{i-1,j} - \psi_{i+1,j}}{2\Delta x}
\end{align*}
\]

3. Calculate vorticity over cavity:

\[
\Omega_{i,j} = \frac{v_{i+1,j} - v_{i-1,j}}{2\Delta x} - \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y}
\]
Computational Procedure

1. Initialize all variables corresponding to \( t = 0 \)
2. Calculate velocities inside the cavity

\[
\begin{align*}
    u_{i,j} &= \frac{\psi_{i,j+1} - \psi_{i,j-1}}{2\Delta y} \\
    v_{i,j} &= \frac{\psi_{i-1,j} - \psi_{i+1,j}}{2\Delta x}
\end{align*}
\]

3. Calculate vorticity over cavity:

\[
\Omega_{i,j} = \frac{v_{i+1,j} - v_{i-1,j}}{2\Delta x} - \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y}
\]
Computational Procedure

1. Initialize all variables corresponding to $t = 0$
2. Calculate velocities inside the cavity

$$u_{i,j} = \frac{\psi_{i,j+1} - \psi_{i,j-1}}{2\Delta y}$$

$$v_{i,j} = \frac{\psi_{i-1,j} - \psi_{i+1,j}}{2\Delta x}$$

3. Calculate vorticity over cavity:

$$\Omega_{i,j} = \frac{v_{i+1,j} - v_{i-1,j}}{2\Delta x} - \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y}$$
Computational Procedure Continued

4. Calculate vorticities

\[
\frac{\Omega_{i,j}^{n+1} - \Omega_{i,j}^n}{\Delta t} + u_{i,j}^n \frac{\Omega_{i+1,j}^n - \Omega_{i-1,j}^n}{2\Delta x} + v_{i,j}^n \frac{\Omega_{i,j+1}^n - \Omega_{i,j-1}^n}{2\Delta y} = \]

\[
= \frac{1}{Re_\infty} \left[ \frac{\Omega_{i+1,j}^n - 2\Omega_{i,j}^n + \Omega_{i-1,j}^n}{(\Delta x)^2} + \frac{\Omega_{i,j+1}^n - 2\Omega_{i,j}^n + \Omega_{i,j-1}^n}{(\Delta y)^2} \right]
\]

5. Solve Poisson’s equation

\[
\tilde{\psi}_{i,j}^{k+1} = \frac{(\Delta x)^2 \Omega_{i,j}^k + \psi_{i+1,j}^k + \psi_{i-1,j}^k + \beta^2 \left( \psi_{i,j+1}^k + \psi_{i,j-1}^k \right)}{2(1 + \beta^2)}
\]

\[
\psi_{i,j}^{k+1} = \omega \psi_{i,j}^k + (1 - \omega) \tilde{\psi}_{i,j}^{k+1}
\]

\[
\beta = \frac{\Delta x}{\Delta y}, \quad u_{i,j} = \frac{\psi_{i+1,j}^k - \psi_{i,j}^k}{2\Delta y}, \quad v_{i,j} = -\frac{\psi_{i,j+1}^k - \psi_{i,j}^k}{2\Delta x}
\]
4. Calculate vorticities

\[ \frac{\Omega_{i,j}^{n+1} - \Omega_{i,j}^{n}}{\Delta t} + u_{i,j}^{n} \frac{\Omega_{i+1,j}^{n} - \Omega_{i-1,j}^{n}}{2\Delta x} + v_{i,j}^{n} \frac{\Omega_{i,j+1}^{n} - \Omega_{i,j-1}^{n}}{2\Delta y} = \]

\[ = \frac{1}{Re_{\infty}} \left[ \frac{\Omega_{i+1,j}^{n} - 2\Omega_{i,j}^{n} + \Omega_{i-1,j}^{n}}{(\Delta x)^{2}} + \frac{\Omega_{i,j+1}^{n} - 2\Omega_{i,j}^{n} + \Omega_{i,j-1}^{n}}{(\Delta y)^{2}} \right] \]

5. Solve Poisson’s equation

\[ \bar{\psi}_{i,j}^{k+1} = \frac{(\Delta x)^{2}\Omega_{i,j}^{k} + \psi_{i+1,j}^{k} + \psi_{i-1,j}^{k} + \beta^{2} \left( \psi_{i,j+1}^{k} + \psi_{i,j-1}^{k} \right)}{2(1 + \beta^{2})} \]

\[ \psi_{i,j}^{k+1} = \omega \psi_{i,j}^{k} + (1 - \omega) \bar{\psi}_{i,j}^{k+1} \]

\[ \beta = \frac{\Delta x}{\Delta y}, \quad u_{i,j} = \frac{\psi_{i,j+1} - \psi_{i,j-1}}{2\Delta y}, \quad v_{i,j} = -\frac{\psi_{i+1,j} - \psi_{i,j-1}}{2\Delta x} \]
Computational Procedure Continued

6. Update values of vorticity at the boundaries

\[ \Omega_{i,1} = -2 \cdot \frac{\psi_{i,2}}{(\Delta y)^2}, \quad \Omega_{i,j} = -2 \cdot \frac{\psi_{i-1,j}}{(\Delta x)^2} \]

\[ \Omega_{1,j} = -2 \cdot \frac{\psi_{2,j}}{(\Delta x)^2}, \quad \Omega_{j,i} = -2 \frac{\psi_{j-1,i}}{(\Delta y)^2} - 2 \cdot \frac{u_o}{\Delta y} \]

7. Calculate velocities

\[ u_{i,j}^{n+1} = \frac{\psi_{i,j+1}^{n+1} - \psi_{i,j-1}^{n+1}}{2\Delta y} \]

\[ v_{i,j}^{n+1} = \frac{\psi_{i+1,j}^{n+1} - \psi_{i-1,j}^{n+1}}{2\Delta x} \]

8. Check convergence

\[ \| u^{n+1} - u^n \| < \epsilon_u, \quad \| v^{n+1} - v^n \| < \epsilon_v \]
Computational Procedure Continued

6. Update values of vorticity at the boundaries

\[ \Omega_{i,1} = -2 \cdot \frac{\psi_{i,2}}{(\Delta y)^2}, \quad \Omega_{i,j} = -2 \cdot \frac{\psi_{i-1,j}}{(\Delta x)^2} \]

\[ \Omega_{1,j} = -2 \cdot \frac{\psi_{2,j}}{(\Delta x)^2}, \quad \Omega_{j,i} = -2 \cdot \frac{\psi_{j-1,i}}{(\Delta y)^2} - 2 \cdot \frac{u_o}{\Delta y} \]

7. Calculate velocities

\[ u_{i,j}^{n+1} = \frac{\psi_{i,j+1}^{n+1} - \psi_{i,j-1}^{n+1}}{2 \Delta y} \]

\[ v_{i,j}^{n+1} = \frac{\psi_{i-1,j}^{n+1} - \psi_{i+1,j}^{n+1}}{2 \Delta x} \]

8. Check convergence

\[ ||u^{n+1} - u^n|| < \epsilon_u, \quad ||v^{n+1} - v^n|| < \epsilon_v \]
Computational Procedure Continued

6. Update values of vorticity at the boundaries

\[ \Omega_{i,1} = -2 \cdot \frac{\psi_{i,2}}{(\Delta y)^2}, \quad \Omega_{i,j} = -2 \cdot \frac{\psi_{i-1,j}}{(\Delta x)^2} \]
\[ \Omega_{1,j} = -2 \cdot \frac{\psi_{2,j}}{(\Delta x)^2}, \quad \Omega_{j,i} = -2 \frac{\psi_{j-1,i}}{(\Delta y)^2} - 2 \cdot \frac{u_0}{\Delta y} \]

7. Calculate velocities

\[ u_{i,j}^{n+1} = \frac{\psi_{i,j+1}^{n+1} - \psi_{i,j-1}^{n+1}}{2\Delta y} \]
\[ v_{i,j}^{n+1} = \frac{\psi_{i-1,j}^{n+1} - \psi_{i+1,j}^{n+1}}{2\Delta x} \]

8. Check convergence

\[ ||u^{n+1} - u^n|| < \epsilon_u, \quad ||v^{n+1} - v^n|| < \epsilon_v \]
Results

Contour plot $\psi = \text{const}$, $I = 41$, $J = 31$, $\nu = 0.0025 \ m^2/\text{s}$.

$u_o = 5 \ m/\text{s}$
Results
Results
Results
Results