# Generalized eigenvalue problems and applications in systems and control

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In these lectures we survey a number of generalized eigenvalue problems and related matrix decompositions and discuss their use in problems of dynamical systems and control. The lectures cover four distinct topics :

1. *Orthogonal state-space transformations*
   These transformations lead to condensed state-space forms, which are useful for solving several analysis and design problems. They also lead to economical Kalman filtering implementations.

2. *Eigenvalues and singular values of matrix sequences*
   The corresponding decompositions form the basis for reliable methods for solving Riccati, Lyapunov and Sylvester equations of periodic systems and also play a role in two point boundary value problems.

3. *Stability radii of dynamical systems*
   These are robustness measures that indicate how much a dynamical system can be perturbed before losing its stability. They are related to the $H_\infty$-norm and $\mu$-norm of a dynamical system.

4. *Model reduction of large-scale dynamical systems*
   Large-scale models often need to be approximated for reasons of computational efficiency. We present Krylov based methods and discuss their efficiency for sparse large-scale dynamical systems.

Each of these topics have received a lot of attention in the last decade and we try to give the basic ideas of the numerical techniques that are currently available. We focus on problems of accuracy, sensitivity and complexity of the relevant algorithms.

# 1 Orthogonal state-space transformations

Canonical forms of state-space models of multivariable linear systems have often been proposed for solving certain analysis and design problems encountered in linear system theory. We show that for many problems one can as well make use of so-called *condensed forms*, which can be obtained under orthogonal state-space transformations.

It is shown that the use of these forms leads to elegant and numerically reliable methods for solving several problems in state-space and generalized state-space systems, including :

- controllability, observability, minimality of state-space and generalized state space models,
- solution of Lyapunov, Sylvester and Riccati equations,
- pole placement and robust feedback,
- observer design,
- frequency response and system response calculations,
- Kalman filtering.

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# 2 Eigenvalues and singular values of matrix sequences

In this lecture we present a unitary eigendecomposition for a sequence of matrices which we call the *periodic Schur decomposition.* We prove its existence and discuss its application to the solution of periodic difference equations arising in control. We show how the classical $QR$ algorithm can be extended to provide a stable algorithm for computing this generalized decomposition. We briefly mention the existence of a similar decomposition and algorithm for an orthogonal decomposition for the singular values of a matrix product. We discuss the definitions, properties and algorithms for these decompositions and we also show their use in several basic problems from various applications :

- calculation of eigenvalues of cyclic matrices,
- solution of periodic Lyapunov and Riccati equations,
- solution of two point boundary value problems with periodic coefficients,
- solution of periodic pole placement.

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# 3 Stability radii of dynamical systems

The basic stability radius problem can be defined as follows. Let $A$ be a *stable* matrix with all its eigenvalues in the stability region $\mathcal{S}$ (typically the open left half plane or the open unit disc). The *stability radius* of $A$ is the norm of smallest (complex) perturbation $\Delta$ that causes at least one eigenvalue of $A+\Delta$ to become unstable, i.e.

$$r_{\mathcal{C}} = \min \left\{ \|\Delta\|_2, \Delta \in \mathcal{C}^{n\times n}, \exists \lambda(A+\Delta) \notin \mathcal{S} \right\}$$

This problem has extensively been studied and there exists an analytic formulation for $r_{\mathcal{C}}$ as well as a quadratically convergent algorithm to compute it. We discuss several variants and extensions of this standard problem to :

- the generalized eigenvalue problem,
- polynomial matrices,
- periodic eigenvalue problems,
- structured matrices.

We discuss the use of different norms as well as structured perturbations and link this problem to the concept of pseudospectra. We will also briefly talk about computational aspects.

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# 4 Model reduction of large-scale dynamical systems

The non-symmetric Lanczos method has received a lot of attention lately as a model reduction technique for large-scale systems of the type :

$$\left\{ \begin{array}{rcl} E\dot{x} & = & Ax + Bu \\ y & = & Cx + Du \end{array} \right. , \tag{1}$$

where the matrices, $A, B, C, D$ and $E$ are sparse and large (i.e. the state dimension $N$ is large). They arise, for example, out of finite-element discretizations of various plants including chemical processes, electronic devices and mechanical systems. Conventional model reduction techniques such as balanced realization, Hankel norm approximations, are ill-suited for large, sparse problems due to the sheer size of the matrices. As an alternative to such model reduction techniques, an oblique Krylov projector can be constructed to produce a lower order model (of state dimension $n << N$) which, driven by the same input $u$, will produce a good appoximation $\hat{y}$ to the output $y$ :

$$\left\{ \begin{array}{rcl} \hat{E}\dot{\hat{x}} & = & \hat{A}\hat{x} + \hat{B}u \\ \hat{y} & = & \hat{C}\hat{x} + Du \end{array} \right. . \tag{2}$$

Here $\hat{A} \doteq Z_n^T A V_n$, $\hat{B} \doteq Z_n^T B$, $\hat{C} \doteq C V_n$, $\hat{E} \doteq Z_n^T E V_n$, and the matrices $V_n \in \mathbb{R}^{N\times n}$ and $Z_n \in \mathbb{R}^{N\times n}$ are related to two Krylov spaces derived from $\{A, E, B\}$ and $\{A, E, C\}$. We show how to incorporate more than one interpolation point into this scheme and explain what Galerkin conditions are met. We also show how the Lanczos algorithm recurrence can estimate the frequency response of a large system and at the same time give an error estimate at a reasonable cost.

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