Dr. Tobias Neckel
Dr. Florian Rupp

Dynamical Systems & Scientific Computing: Homework Assignments

1.1 [☼] Finite Differences in 1D

a) Derive the following 3 finite difference (FD) approximations for the first order derivative of a 1D function $u(x) \in C^3(x-h, x+h)$:

$$\partial^+ u(x) \;:=\; \frac{u(x+h) - u(x)}{h} \,, \tag{1}$$

$$\partial^- u(x) \;:=\; \frac{u(x) - u(x-h)}{h} \,, \tag{2}$$

$$\partial^0 u(x) \;:=\; \frac{u(x+h) - u(x-h)}{2h} \,. \tag{3}$$

What is the corresponding order of approximation of each variant (1)–(3)? Why?
Hint: Use the Taylor expansions of $u(x \pm h)$.

b) A FD approximation for the second order derivative of the 1D function $u(x) \in C^4(x-h, x+h)$ is defined in the following manner:

$$\partial^- \partial^+ u(x) \;:=\; \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} \,. \tag{4}$$

- Derive (4) via the Taylor expansion. What is the order of approximation?
- Derive (4) in a direct way using (1)–(3).

1.2 [⊕] Finite Differences in 2D

We consider the two-dimensional Poisson Equation

$$\Delta u(x,y) \;=\; -\sin(\pi x) \cdot \sin(\pi y) \cdot 2\pi^2 \,, \qquad (x,y) \in \Omega := [0,1]^2 \,, \tag{5}$$

$$u(x,y) \;=\; 0 \,, \qquad (x,y) \in \Gamma := \partial\Omega \,, \tag{6}$$

where $\Delta u(x,y)$ represents the Laplacian $\Delta u(x,y) = \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y)$.

a) Apply the 1D FD approximations of Exercise **1.1** to derive a FD approximation for the 2D Laplacian $\Delta u(x,y)$.

b) Let

$$\Omega_h \;:=\; \{(ih, jh) \mid i = 1, \ldots, N, j = 1, \ldots, N\} \tag{7}$$

be a mesh of N inner grid points in each direction on the unit square $[0,1]^2$ for a given mesh size $h_x = h_y = h := 1/(N+1)$.

- Draw a sketch how the mesh Ω_h looks like.
- Which neighbouring points are necessary in order to evaluate the FD approximation of exercise a) for a given point (i,j)?
- Sketch the matrix A of the linear system of equations $Au = b$ that is created by applying the FD of Exercise a) to the given problem (5)–(6) using the mesh (7).

c) Write a short matlab script which implements the numerical solution of (5)–(6).
Hint: Use the matlab backslash operator ("\") to solve the linear system of equations.
How could you verify that your implementation is correct?

1.4 [☼] Recalling the Basics of Probability Theory – pt. 1

Suppose X is a random variable having the probability density function

$$f(x) = \begin{cases} R \cdot x^{R-1} & \text{for } 0 \leq x \leq 1, \\ 0 & \text{elsewhere,} \end{cases}$$

where $R > 0$ is a fixed parameter.

a) Determine the distribution function $F_X(x)$.

b) Determine the mean $\mathbb{E}(X)$.

c) Determine the variance $\mathbb{V}ar(X)$.

1.5 [☼] Recalling the Basics of Probability Theory – pt. 2

A special piece of control hardware for the ARIANE 5 rocket consists of two main components: A and B. The operating times until failure of the two components are independent and exponentially distributed random variables with parameter 2 for component A, and 3 for B. The system fails at the first component failure making it impossible to land the ARIANE rocket safely.

a) What is the mean time to failure for component A? For component B?

b) What is the mean time to system failure?

c) What is the probability that it is component A that causes the system failure?

d) Suppose that it is component A that fails first. What is the mean remaining operating life of component B?

Classification: ☼ easy, ◷ easy with longer calculations, ✰ a little bit difficult, ⧉ challenging.