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Dynamical Systems & Scientific Computing: Homework Assignments

2.1 [☼] **(Solution of an Inhomogeneous Linear ODE-System)**
Give the general real solution of the following two-dimensional inhomogeneous system of linear differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix},$$

where $x, y : \mathbb{R} \to \mathbb{R}$ and $t \in \mathbb{R}$.

2.2 [☼] **(Stability via Lyapunov-Functions)**

a) Show that the zero solution of the system

$$\dot{x} = -x - xy^2, \qquad \dot{y} = -y - x^2 y,$$

is globally asymptotically stable, by guessing a suitable Lyapunov function.

b) Investigate the stability of the zero solution of

$$\dot{x} = -xy - x, \qquad \dot{y} = y^3 - xy^3 + xy + y,$$

by using the function $V(x, y) = -x - \ln(1 - x) - y - \ln(1 - y)$ locally around $(x, y) = (0, 0)$.

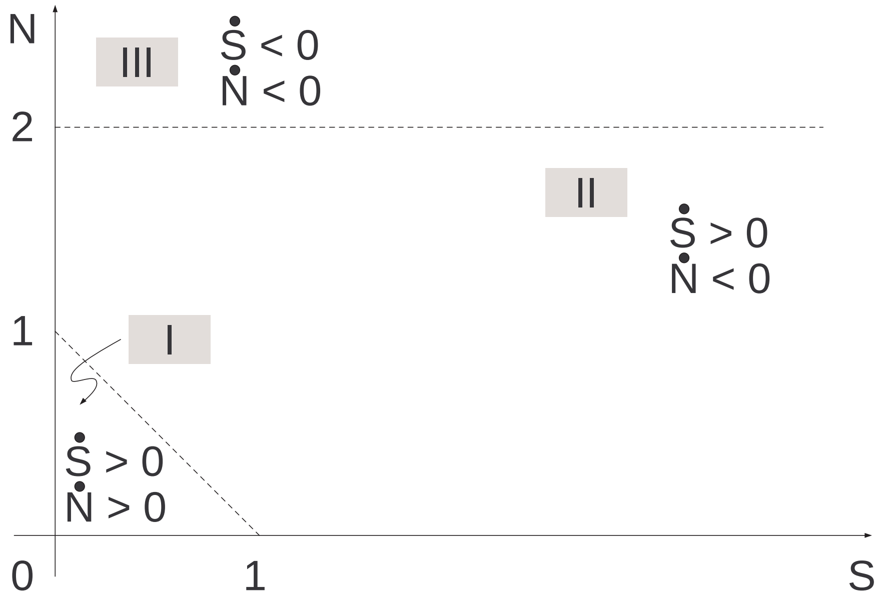
2.3 [☼]-[◷] **(What Really Happened at the Paris Peace Talks)**
The original plan developed by Henry Kissinger and Le Duc Tho to settle the Vietnamese war is described below. It was agreed that 1 million South Vietnamese ants and 1 million North Vietnamese ants would be placed in the backyard of the Presidential palace in Paris and be allowed to fight it out for a long period of time. If the South Vietnamese ants destroyed nearly all the North Vietnamese ants, then South Vietnam would retain control of all of its land. If the North Vietnamese ants were victorious, then North Vietnam would take over all of South Vietnam. If they appeared to be fighting to a standoff then South Vietnam would be partitioned according to the proportion of ants remaining. Now, the South Vietnamese ants, denoted by S, and the North Vietnamese Ants, denoted by N, compete against each other according to the following differential equations:

$$\frac{dS}{dt} = \frac{1}{10}S - \frac{1}{20}S \times N \qquad (1)$$

$$\frac{dN}{dt} = \frac{1}{100}N - \frac{1}{100}N^2 - \frac{1}{100}S \times N \qquad (2)$$

Note that these equations correspond to reality since South Vietnamese ants multiply much more rapidly than the North Vietnamese ants, but the North Vietnamese ants are much better fighters. The battle began at 10:00 sharp on the morning of May 19, 1972, and was supervised by a representative of Poland and a representative of Canada. At 2:43 p.m. on the afternoon May 21, the representative of Poland , being unhappy with the progress of the battle, slipped a bag of North Vietnamese ants into the backyard, but he was spotted by the eagle eyes of the representative of Canada. The South Vietnamese immediately claimed a foul and called off the agreement, thus setting the stage for the protracted talks that followed in Paris. The representative of Poland was hauled before a judge in Paris for sentencing. The judge, after making some remarks about the stupidity of the South Vietnamese gave the Polish representative a very light sentence. Justify mathematically the judges's decision.

Hint:



a) Show that the lines $N = 2$ and $N + S = 1$ divide the first quadrant into three regions (see the figure) in which dS/dt and dN/dt have fixed signs.

b) Show that every solution $S(t)$, $N(t)$ of (1)-(2) which start in region I or region III must eventually enter region II.

c) Show that every solution $S(t)$, $N(t)$ of (1)-(2) which start in region II must remain there for all future time.

d) Conclude from (c) that $S(t) \to \infty$ for all solutions $S(t)$,$N(t)$ of (1)-(2) with $S(t_0)$ and $N(t_0)$ positive. Conclude, too, that $N(t)$ has a finite limit (≤ 2) as $t \to \infty$.

e) To prove that $N(t) \to 0$, observe that there exists t_0 such that $dN/dt \leq -N$ for $t \geq t_0$. Conclude from these inequality that $N(t) \to 0$ as $t \to \infty$.

2.4 [☼] **(Simulating Solutions and Phase Space Diagrams)**
For $t \in \mathbb{R}$ consider the forced linear oscillator differential equation

$$\ddot{x}(t) + 2cd\dot{x}(t) + d^2x(t) \;=\; f(t)\,, \qquad x \in \mathcal{C}^2(\mathbb{R}, \mathbb{R})\,,$$

with $f(t) = e^{-0.1t}\sin(10t)$ and constants $c = 0.64$ and $d = 15.56$. In the case that f would be Gaussian white noise, this equation actually models an earthquake excitation for firm soil conditions according to Kanai and Tajimi's work.

a) Solve this oscillator equation numerically with the explicit Euler method for the initial condition $x(0) = 3$ and step sizes of 10^{-2}, 10^{-4} as well as 10^{-6}. Plot the solution against the time variable $t \in [0, 10]$, and in the x-$\dot{x}$-phase plane.

b) Solve this oscillator equation numerically with the implicit Euler method for the initial condition $x(0) = 3$ and step sizes of 10^{-2}, 10^{-4} as well as 10^{-6}. Plot the solution against the time variable $t \in [0, 10]$, and in the x-$\dot{x}$-phase plane.

c) As usual, the energy of this oscillator is given by

$$E(\dot{x}, x, t) \;=\; \tfrac{1}{2}\,(\dot{x}(t))^2 + \tfrac{1}{2}d^2\,(x(t))^2\ .$$

Compare the energies obtained for the numerical solutions by the explicit and implicit Euler method.

We will extend this little model equation in the following exercises to allow a reasonable treatment of earthquake induced ground motion effects on multi-story buildings.

Classification: ☼ easy, ◷ easy with longer calculations, ✩ a little bit difficult, ⧉ challenging.