

# PSE Game Physics

Session (6)

Angular momentum, microcollisions, damping

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*TUM Uhrenturm*

# Outline

## Angular momentum

- Angular velocity - starting point

- Torque

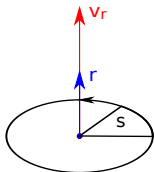
- Angular momentum

## Angular velocity - starting point

- With our quaternions, we are able to setup the rotation of an object.
- Question: **How to express the angular velocity in 3D?**
- Describe the angular velocity  $\vec{v}_r$  by a simple vector:

$$\vec{v}_r = s \cdot \vec{r}$$

that is a product of the norm of the angular speed  $s$  and the normalized axis of rotation  $\vec{r}$ .



- Remark:  $s$  corresponds to the time derivative  $\dot{\theta}$  of the rotation angle  $\theta$  and has the unit angle per second.

# Angular velocity - multiple velocities

- Translational velocities are additive under composition:

$$\vec{v} = \vec{v}_1 + \vec{v}_2$$

- Angular velocities are additive under composition, too (although the explanation is not trivial):

$$\vec{v}_r = \vec{v}_{r1} + \vec{v}_{r2}$$

## Angular velocity - from acceleration to velocity

- For the translational acceleration, we used the **explicit Euler** to approximate the updated velocity:

$$\vec{v}(t + \Delta t) = \vec{v}(t) + \Delta t \cdot \vec{a}$$

and an **implicit Euler** to update the position of the object (using the *translate()* method in the engine):

$$\vec{x}(t + \Delta t) = \vec{x}(t) + \Delta t \cdot \vec{v}(t + \Delta t)$$

- For a given angular acceleration  $\vec{\alpha}$ , we can update the angular velocity with a similar method:

$$\vec{v}_r(t + \Delta t) = \vec{v}_r(t) + \Delta t \cdot \vec{\alpha}$$

What about updating the rotation?

$$q(t + \Delta t) = ???$$

# From angular velocity to quaternions

- Remember Session (3):
- Given rotation axis  $(x, y, z)$  and angle  $\theta = |\vec{v}_r|\Delta t$

$$\rightarrow q_r := \begin{pmatrix} w \\ i \\ j \\ k \end{pmatrix} = \begin{pmatrix} \cos \frac{\theta}{2} \\ x \sin \frac{\theta}{2} \\ y \sin \frac{\theta}{2} \\ z \sin \frac{\theta}{2} \end{pmatrix} \in \mathbb{R} \times \mathbb{R}^3$$

- Computing a single timestep given an angular velocity vector  $\vec{v}_r$ , a quaternion can be set up representing the rotation for a single timestep.
- Thus, you are able to compute the new quaternion of the next timestep:

$$q(t + \Delta t) = q(t) \cdot q_r(\Delta t)$$

# Moving on...

- So far we (should) know:
  - What an angular velocity vector is.
  - How to rotate an object during each timestep.
    - Update angular velocity of objects using angular acceleration.
    - Update the orientation by converting the angular rotation to a quaternion and applying the rotational quaternion to the one describing the orientation of the object.
- Next: How to modify the angular velocity of an object OR how to obtain  $\vec{\alpha}$ ?

# Torque

(Drehmoment)



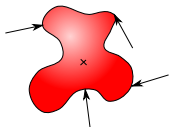
**Figure :** Force acting on object

- Let's assume to have a directed force  $\vec{F}$  (e. g. created by a spring) exceeded on the point  $\vec{x}'$  relative to the center of mass of an object.
- Then the torque is computed by

$$\vec{\tau} = \vec{x}' \times \vec{F}$$



# Multiple Torques



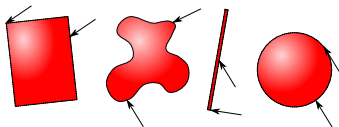
**Figure :** Multiple forces acting on an object

- We are not only interested in a single torque acting on an object
- When multiple forces are exerted on an object, they have to be combined somehow
- Similar to forces, torques can be added up:

$$\vec{\tau} = \sum_i \vec{x}'_i \times \vec{F}_i$$

# Inertia

(Massenträgheit)



- Change of angular velocity due to torque clearly depends on the **shape of the object!**
- This information can be expressed in form of the **inertia tensor**.
- For spheres and boxes, these tensors are

$$I_{\text{Sphere}} = \begin{pmatrix} \frac{2}{5}mr^2 & \cdot & \cdot \\ \cdot & \frac{2}{5}mr^2 & \cdot \\ \cdot & \cdot & \frac{2}{5}mr^2 \end{pmatrix} \quad I_{\text{Box}} = \begin{pmatrix} \frac{1}{12}m(w^2 + d^2) & \cdot & \cdot \\ \cdot & \frac{1}{12}m(h^2 + d^2) & \cdot \\ \cdot & \cdot & \frac{1}{12}m(h^2 + w^2) \end{pmatrix}$$

See [http://en.wikipedia.org/wiki/List\\_of\\_moments\\_of\\_inertia](http://en.wikipedia.org/wiki/List_of_moments_of_inertia) for more inertia tensors

## Inertia (1/2)

- Similar to the translational force  $\vec{F} = m \cdot \vec{a}$ , we get a formula for the torque depending on the angular acceleration  $\alpha$  and the inertia tensor:

$$\vec{\tau} = I\vec{\alpha}$$

- Since we are **interested in computing the change in the angular acceleration**, we are allowed to rewrite the equation:

$$\vec{\alpha} = I^{-1}\vec{\tau}$$

- **Warning:** This formula is valid only in object space!!!  
However, the forces exerted on the object might only be given in world space.

⇒ convert to world space

## Inertia (2/2)

- The torque  $\tau$  is given as a **vector**
- Object  $\rightarrow$  World space: inverse transposed matrix (see session (2))
- World  $\rightarrow$  Object space: transposed matrix
- Putting everything together, the angular acceleration vector is computed by

$$\vec{\alpha} = M^{-T} I^{-1} M^T \vec{\tau}$$

with model matrix  $M$

# Angular momentum

- We start by looking at our “simple“ **translational case**. There we computed the momentum by

$$\vec{p}_l = m \cdot \vec{v}$$

- For angular rotations, we express the **angular momentum** in a similar way:

$$\vec{p}_r = I \cdot \vec{v}_r$$

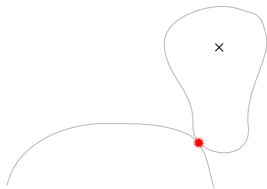
- The angular momentum change created by a **translational momentum** change at point  $\vec{x}'$  on the object is given by

$$\Delta \vec{p}_r = \vec{x}' \times \Delta \vec{p}_l$$

- Finally we can express the change of angular velocity due to momentum by

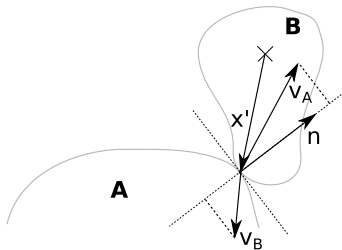
$$\Delta \vec{v}_r = I^{-1} \Delta \vec{p}_r = I^{-1} (\vec{x}' \times \Delta \vec{p}_l)$$

# Angular collision impulse



- From the translational case, we know that due to the collision, the “velocity has to change“ to match  $v_s = -C_r \cdot v_c$  and other physical properties.
- In particular, the **velocity of both collision points** has to match this formula.
- Our goal is to **compute the impulse which has to be applied to the collision points** to match  $v_s$ .

## Step 1) Compute the closing velocity

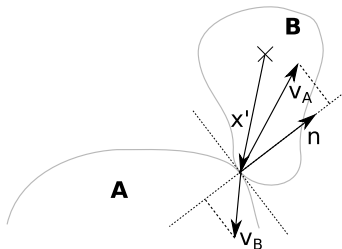


- The velocity of a point on the surface of the object is given by

$$\vec{v} = \vec{v}_r \times \vec{x}' + \vec{v}_l$$

with *angular velocity*  $\vec{v}_r$ ,  $\vec{x}'$  as the vector from the center of mass to the collision point and  $\vec{v}_l$  the (*linear*) *translational velocity*.

## Step 1) Compute the closing velocity (con'd)



- Since **only the velocity towards the collision normal** may be modified (without friction!), this fraction is computed by using the dot product:

$$v_c = \vec{n} \cdot \vec{v}_A - \vec{n} \cdot \vec{v}_B$$

- Now we are able to compute the separating velocity with the same formulas as for the purely translational case.



## Step 2) Compute the change in velocity for unit impulse

- It's problematic to rearrange the formulas to directly compute the amount of impulse which has to be applied to gain a specific separating velocity.
- Therefore we utilize an important property of the impulse (similar to Newtons 3rd law):  
*When an impulse is exceeded to an object during a collision, an impulse of same magnitude but opposite direction is exceeded to the other object.*
- **Important:** The direction of the unit impulse of one object aims to the opposite direction of the unit impulse applied to the other object.
- The change in velocity is computed based on applying a **unit impulse** to both objects which is explained successively in the next slides.

## Step 2) Compute the change in velocity for unit impulse (con'd)

- The **unit impulse is parallel to the collision normal**.
- Computing the change in translational velocity for the unit impulse is/(should be) known from previous sessions:

$$\Delta \vec{v}_l = \vec{n} \cdot m^{-1}$$

- The change in angular velocity is computed using

$$\Delta \vec{v}_r = \underbrace{M^{-T} I^{-1} M^T}_{\text{Inertia to world space}} \left( \underbrace{\vec{x}'}_{\text{lever arm}} \times \underbrace{\vec{n}}_{\text{unit impulse}} \right)$$

- Finally, we compute the change of velocity in the specific point on the object:

$$\Delta \vec{v} = \Delta \vec{v}_l + \Delta \vec{v}_r \times \vec{x}'$$

- This has to be computed for both objects!

## Step 3) Computing the impulse to apply (1/3)

- From step 1), we know the **closing velocity**:

$$v_c = \vec{n} \cdot \vec{v}_A - \vec{n} \cdot \vec{v}_B$$

- With the coefficient of restitution, we can compute the **separating velocity** using the formula:

$$C_r = -\frac{v_s}{v_c}$$

- From step 2), we can also compute the **change of velocity in the direction of the normal** for the **unit impulse**:

$$\Delta s = \Delta \vec{v} \cdot \vec{n}$$

## Step 3) Computing the impulse to apply (2/3)

- The difference of the separating and closing velocities from step 1) is obviously not equal to the change of velocities was computed with the unit impulse.
- However, it is equal to a **fraction**  $f$  of the unit impulse and since all operations are linear:

$$v_s - v_c = f \cdot (\Delta s_A + \Delta s_B)$$

(Depending on your implementation, maybe you have to change the sign of one component!)

## Step 3) Computing the impulse to apply (3/3)

- Current state: We know the fraction of the unit impulse which has to be applied to both objects to gain the separating velocity!
- There are 2 ways how to apply this impulse:
  - Either we apply the impulse which is now well known by multiplying the fraction  $f$  with the unit impulse which has to be applied to gain the separating velocity or
  - we use directly the fraction  $f$  to compute the change of translational and angular velocity for each object by using our **previously computed variables**  $\Delta \vec{v}_l$  and  $\Delta \vec{v}_r$ .

$$\vec{v}_l = \vec{v}_l + f \cdot \Delta \vec{v}_l$$

$$\vec{v}_r = \vec{v}_r + f \cdot \Delta \vec{v}_r$$

## Summary (1/2)

Steps to resolve a collision:

- Choose as (initially wrong) impulse the unit vector  $\vec{n}$  (which is the collision normal).
- For each object, compute the change of linear and rotation velocity ( $\Delta\vec{v}_l$  and  $\Delta\vec{v}_r$ ) in the collision point, that would occur if we applied the unit impulse  $\vec{n}$  (slide 18).
- For each object, compute the total velocity change  $\Delta\vec{v}$  at the collision point by adding up linear velocity and cross product of rotation velocity and lever arm:

$$\Delta\vec{v} = \Delta\vec{v}_l + \Delta\vec{v}_r \times \vec{x}'$$

## Summary (2/2)

Compute a reasonable fraction  $f$  of the unit impulse:

- Applying the unit impulse to two colliding objects, we would get the separating velocity:

$$v_s = v_c + (\Delta \vec{v}_A - \Delta \vec{v}_B) \cdot \vec{n}$$

However,  $v_s$  does not fulfill the condition (slide 19):

$$v_s = -C_r \cdot v_c$$

- So in order to fulfill the condition, we have to multiply the unit impulse with a fraction  $f$ . Since all performed operations are linear, we get:

$$-C_r \cdot v_c = v_s = v_c + f \cdot (\Delta \vec{v}_A - \Delta \vec{v}_B) \cdot \vec{n}$$

- Solve the formula by  $f$  and set:

$$\vec{v}_l = \vec{v}_l + f \cdot \Delta \vec{v}_l$$

$$\vec{v}_r = \vec{v}_r + f \cdot \Delta \vec{v}_r$$

## Some more hints & comments:

- Always consider the **different spaces** (model or world space) in which you're doing the computations. Sometimes you'll have to do some transformations.
- To gain better results, also the **interpenetration** has to account for the **rotations** of objects.