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Practical Course Scientific Computing and Visualization

Worksheet 4

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In the last worksheet, we examined the stationary heat equation, that is the temperature T was depending on space only. Now, we generalize to the instationary heat equation

$$T_t = T_{xx} + T_{yy} \tag{1}$$

on the unit square $]0;1[^2$ with the temperature T, the two-dimensional coordinates x and y, the time t, homogeneous Dirichlet boundary conditions

$$T(x,y,t) = 0 \text{ forall } (x,y) \text{ in } \partial]0;1[^2, t \text{ in }]0;\infty[\tag{2}$$

and

$$T(x,y,0) = 1.0 \text{ forall } (x,y) \text{ in }]0;1[^2 \tag{3}$$

as initial condition.
For the spatial discretization, we again use the grid points $\left\{(x_i,y_j) = \left(i \cdot \frac{1}{h_x}, j \cdot \frac{1}{h_y}\right), i = 0,1,\ldots,N_x,N_x+1, j = 0,1,\ldots,N_y,N_y+1\right\}$ and the finite difference approximation of the second derivatives

$$T_{xx}|_{i,j} \approx \frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{h_x^2},$$

$$T_{yy}|_{i,j} \approx \frac{T_{i,j-1} - 2t_{i,j} + T_{i,j+1}}{h_y^2}$$

for $i = 1,\ldots,N_x, j = 1,\ldots,N_y$ with $h_x = \frac{1}{N_x+1}$ and $h_y = \frac{1}{N_y+1}$.

a) To have a rough insight in the exact solution of (1), (2), and (3), determine

$$\lim_{t\to\infty} T(x,y,t).$$

b) Implement an explicit Euler step for (1) and (2) as a function of the grid sizes N_x and N_y, the time step δt, and the computed solution at the current time.

c) Solve (1), (2), and (3) with the help of this Euler method for the values of δt, N_x, and N_y listed in the tabular below and plot the solutions at times $t = \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}$. Mark cases with stable solutions in the following tabular.

$N_x = N_y$	$\delta t = \frac{1}{64}$	$\delta t = \frac{1}{128}$	$\delta t = \frac{1}{256}$	$\delta t = \frac{1}{512}$	$\delta t = \frac{1}{1024}$	$\delta t = \frac{1}{2048}$	$\delta t = \frac{1}{4096}$
3							
7							
15							
31							

d) Implement an implicit Euler step for (1) and (2) as a function of the grid sizes N_x and N_y, the time step δt, and the computed solution at the current time.

Hint: Use a modification of the Gauss-Seidel solver implemented in worksheet 3 for the solution of the system of linear equations in each timestep. Iterate until the norm of the residual is below 10^{-4}.

e) Solve (1), (2), and (3) with the help of the implicit Euler method and time step $\delta t = \frac{1}{64}$. Plot the solutions at times $t = \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}$. Do you get stable solutions?

Questions:

1) What relation between maximal time step size (to achieve a stable discretization) and spatial step $h_x = h_y$ can you derive from the tabular in **b)**?

2) Would you consider it reasonable to use a higher order explicit time discretization in **b)**?

Hint: Assume that all explicit discretizations have similiar restrictions with respect to time step size as seen above for the explicit Euler method and think about the effort you have to make to get a stable solution with balanced accuracy in time and space!!!

3) Is it reasonable to use an imlicit Euler method in our example if we want to compute solutions with balanced accuracy in time and space for each N_x, N_y?

4) Would you consider it reasonable to use a higher order implicit time discretization (again from the point of view of balanced accuracy in time and space)?

   Hint: Assume that all implicit discretizations have similar stability properties as the implicit Euler method.