Scientific Computing Lab

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Discretisation of Boundary Conditions

In worksheet 3, we solve a two-dimensional stationary heat equation

$$T_{xx} + T_{yy} = f(x, y) \text{ in }]0;1[^2 \tag{1}$$

with the temperatur $T(x, y)$, the two-dimensional coordinates x and y, and homogeneous Dirichlet boundary conditions

$$T(x, y) = 0 \text{ forall } (x, y) \text{ in } \partial]0;1[^2. \tag{2}$$

We discretised this equation on a regular Cartesian grid with vertices $(i \cdot h_x, j \cdot h_y)$ using the 5-point-stencil

$$\begin{bmatrix} & \frac{1}{h_y^2} & \\ \frac{1}{h_x^2} & -\frac{1}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} \\ & \frac{1}{h_y^2} & \end{bmatrix}$$

As all values at the boundaries were zero, the boundary conditions did not explicitely appear in the resulting system of linear equations. We will now consider more general boundary conditions and see how they influence the system of linear equations.

Remenber for all boundary conditions: Values at the boundary may not be integrated in the vector of unknowns. Instead, consider the boundary conditions either by changes in the right hand side or changes of the discretisation stencil at grid verzices neighbouring the boundary.

a) Establish the system of linear equations for non-homogeneous Dirichlet boundary conditions

$$T(x, y) = g(x, y) \text{ forall } (x, y) \text{ in } \partial]0;1[^2. \tag{3}$$

Hint: modify the right-hand side according to the boundary conditions.

b) Establish the system of linear equations for homogeneous Neumann boundary conditions

$$\frac{\partial T(x, y)}{\partial \vec{n}} = 0 \text{ forall } (x, y) \text{ in } \partial]0;1[^2, \tag{4}$$

where $\vec{n}$ denotes the normal vector of the respective boundary.

Hint: modify the stencils at vertices neighbouring the boundary.