Scientific Computing Lab

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Discretisations in Time and Space

In solving worksheet 4, you observe time step restrictions for the explicit Euler method similar to those already observed for ODEs in worksheet 2. Today, we will have an a little closer look at these time step restrictions and its implications for the relation between time and spatial discretisations.

We start with an analysis of a simple ODE:

$$\dot{y} = -\lambda y, y(0) = y_0 \text{ with } \lambda > 0.$$

a) Apply an explicit Euler with a fixed time step dt and try to express $y^{(n)}$ as a function of y_0.

Solution:

$$\begin{aligned}
y^{(0)} &= y_0, \\
y^{(1)} &= y_0 + dt \cdot (-\lambda y_0) = (1 - dt \cdot \lambda) y_0, \\
y^{(2)} &= y^{(1)} + dt(-\lambda y^{(1)} = (1 - dt \cdot \lambda) y^{(1)} = (1 - dt \cdot \lambda)^2 y_0, \\
&\vdots \\
y^{(n)} &= (1 - dt \cdot \lambda)^n y_0.
\end{aligned}$$

b) The exact solution of our ODE is

$$y(t) = y_0 e^{-\lambda t}$$

with

$$\lim_{t \to \infty} y(t) = 0.$$

Which restriction does dt have to fulfill such that also

$$\lim_{n \to \infty} y^{(n)} = 0?$$

Solution:

$$
\begin{aligned}
\lim_{n\to\infty} y^{(n)} = 0 \quad &\Leftrightarrow \quad \lim_{n\to\infty} (1 - dt\cdot\lambda)^n = 0 \\
\Leftrightarrow |1 - dt\cdot\lambda| < 1 \quad &\Leftrightarrow \quad dt\cdot\lambda < 2 \\
\Leftrightarrow dt < \frac{2}{\lambda} &
\end{aligned}
$$

If we now look at the explicit Euler for our system of equations resulting from the spatial discretisation of the instationary heat equation

$$\vec{T}^{(n+1)} = \vec{T}^{(n)} + dt\mathbf{M}\vec{T}^{(n)}$$

with the matrix

$$
\mathbf{M} = \frac{1}{h^2}\begin{pmatrix}
-4 & 1 & & 1 & & & & & \\
1 & -4 & 1 & & 1 & & & & \\
 & \ddots & \ddots & \ddots & & \ddots & & & \\
1 & & 1 & -4 & 1 & & 1 & & \\
 & \ddots & & \ddots & \ddots & \ddots & & \ddots & \\
 & & 1 & & 1 & -4 & 1 & & 1 \\
 & & & \ddots & & \ddots & \ddots & \ddots & \\
 & & & & 1 & & 1 & -4 & 1 \\
 & & & & & 1 & & 1 & -4
\end{pmatrix},
$$

we have a set of linear ODEs. Probably, similar restrictions for the time step hold. If our initial state $\vec{T}^{(0)}$ is an eigenvector of the matrix M for the eigenvalue $-\lambda$, the analytical solution of this system is

$$\vec{T}(t) = \vec{T}^{(0)} e^{-\lambda t}.$$

The eigenvalue $-\lambda$ seems to play the same role here as the scalar λ for the linear ODE above. In fact, a strict analysis of the problem leads to the conclusion, that the time step restriction for our system of equations if solved with an explicit Euler method is

$$dt < \frac{2}{\lambda_{max}}$$

with λ_{max} denoting the absolute value of the smallest (negative) eigenvalue of M. Sine M and its eigenvalues are poportional to $\frac{1}{h^2}$, it turns out that

$$dt < ch^2 \text{ with some constant } c.$$

Can you approve this with the results of worksheet 4?

c) Perform a brainstorming to collect aspects that have to be taken into account when chosing a spatial and a time discretisation. Think of both aspects concerning only one of these discretisations and aspects concerning their interplay.

Possible result:

- What accuracy is required?
  High accuracy required → higher order methods
  Low accuracy requrements → low order method sufficient?
- Do we have a stiff equation/system of equations?
  stiff (system of) equations → implicit time stepping methods
  else → explicit time stepping methods
- Do we have a balanced accuracy between the time and the spatial discretisation?

$$dt^p = c \cdot h^q$$

  if we have a pth order time discretisation and a qth order spatial discretisation.
- Did we take into account the stability restriction for the time step in the case of stiff equations?

$$dt < c \cdot h^2$$

  for explicit Euler and the heat equation, e.g.
- Is the stability restriction a constraint or is the restriction of the time step in dependence of the spatial step anyhow given in order to achieve a balanced accuracy?
  Example: explicit Euler and five-point-stencil for the 2D heat equation:

$$dt = c \cdot h^2$$

  follows both from the balanced accuracy and the stability requirement.
  In this case, replacing the explicit method by an implicit methoid of the same order does not make sense!