Scientific Computing Lab

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Multigrid

We solve the one-dimensional Poisson equation with homogeneous Dirichlet boundary conditions

$$T_{xx} = 0 \text{ in }]0;1[, T = 0 \text{ at } \partial]0;1[$$

using the three-point stencil $\frac{1}{h^2}[1 - 21]$ as a discretisation on a regular grid with grid points $x_i = i \cdot h, i = 1, \ldots, N$.

The initial guess is $T_i = 1$ for all $i = 1, \ldots, N$.

What sould be shown in advance:

- red-black Gauss-Seidel.

What sould be the gained knowledge:

- recapitulation of the multigrid method, this time **with** recursive call on the coarse grid,
- initial guess on the coarse grid is always zero.
- injection doesn't always work as a restriction (for the xamples in Worksheet 5, it does work),
- (how to perform bilinear interpolation in 2d (explain separately)).

Hint: Add graphs of the solution, the residuals, and the error approximations evolving during the multigrid cycle on the different grid levels.

For $h = \frac{1}{8}$, we apply a multigrid method with the following steps:

1) Red-black Gauss-Seidel (1 iteration):

   red:

$$\begin{aligned} T_1 = T_7 &= \frac{1}{2}(0+1) = \frac{1}{2}, \\ T_3 = T_5 &= \frac{1}{2}(1+1) = 1. \end{aligned}$$

black:

$$
\begin{aligned}
T_2 = T_6 &= \frac{1}{2}\left(\frac{1}{2}+1\right) = \frac{3}{4}, \\
T_4 &= \frac{1}{2}(1+1) = 1.
\end{aligned}
$$

2) compute the residual:

$$
\begin{aligned}
r_i &= 0 - 8^2(T_{i-1} - 2T_i + T_{i+1}), \\
r_1 = r_7 &= -64\left(0 - 2\frac{1}{2} + \frac{3}{4}\right) = 16, \\
r_2 = r_6 &= -64\left(\frac{1}{2} - 2\frac{3}{4} + 1\right) = 0, \\
r_3 = r_5 &= -64\left(\frac{3}{4} - 2 + 1\right) = 16,
\end{aligned}
$$

$$r_4 = -64(1 - 2 + 1) = 0.$$

3) restrict the residual:

If we would apply simple injection, the residual on the coarse grid would become zero and, thus, also the coarse grid solution would be zero $\Rightarrow$ no coarse grid corection!

Therefore, we apply a better restriction operator, the full-weighting:

$$
\begin{aligned}
rhs_i &= \frac{1}{4}r_{2i-1} + \frac{1}{2}r_{2i} + \frac{1}{4}r_{2i+1}, \\
rhs_1 = rhs_2 = rhs_3 &= \frac{1}{4}16 + \frac{1}{2}0 + \frac{1}{4}16 = 8.
\end{aligned}
$$

4) initial guess on the coarse grid:

$$\bar{E}_1 = \bar{E}_2 = \bar{E}_3 = 0.$$

5) Red-black Gauss-Seidel (1 iteration):

red:

$$\bar{E}_1 = \bar{E}_3 = \frac{1}{2}(0+0) - \frac{1}{4^2}\cdot 8 = -\frac{1}{2}.$$

black:

$$\bar{E}_2 = \frac{1}{2}\left(-\frac{1}{2} - \frac{1}{2}\right) - \frac{1}{4^2}\cdot 8 = -1.$$

6) compute the residual:

$$\begin{aligned}\bar{r}_i &= rhs_i - 4^2(\bar{E}_{i-1} - 2\bar{E}_i + \bar{E}_{i+1}),\\ \bar{r}_1 = \bar{r}_3 &= 8 - 16\left(0 + 2\frac{1}{2} - 1\right) = 8,\\ \bar{r}_2 &= 8 - 16\left(-\frac{1}{2} + 2 - \frac{1}{2}\right) = -8.\end{aligned}$$

7) restrict the residual (full weighting):

$$r\bar{h}s_1 = \frac{1}{4}\bar{r}_1 + \frac{1}{2}\bar{r}_2 + \frac{1}{4}\bar{r}_3 = 2 - 4 + 2 = 0.$$

Thus, we will get an equation with zero right hand side and zero boundary conditions on the coarsest grid. Therefore, we can immediately return to the grid with $h = \frac{1}{4}$.

8) Red-black Gauss-Seidel (1 iteration):

red:

$$\bar{E}_1 = \bar{E}_2 = \frac{1}{2}(0-1) - \frac{1}{4^2}8 = -1.$$

black:

$$\bar{E}_2 = \frac{1}{2}(-1-1) - \frac{1}{4^2}8 = -\frac{3}{2}.$$

9) Interpolate $\bar{E}$ to the fine grid:

$$\begin{aligned}E_1 = E_7 &= \frac{1}{2}(0-1) = -\frac{1}{2},\\ E_2 = E_6 &= -1,\\ E_3 = E_5 &= \frac{1}{2}\left(-1 - \frac{3}{2}\right) = -\frac{5}{4},\\ E_4 &= -\frac{3}{2}.\end{aligned}$$

10) Apply the correction:

$$\begin{aligned}T_i &= T_i + E_i, \text{ for all } i = 1, \ldots, 7,\\ T_1 = T_7 &= \frac{1}{2} - \frac{1}{2} = 0,\\ T_2 = T_6 &= \frac{3}{4} - 1 = -\frac{1}{4},\\ T_3 = T_5 &= 1 - \frac{5}{4} = -\frac{1}{4},\\ T_4 &= 1 - \frac{3}{2} = -\frac{1}{2}.\end{aligned}$$

11) Red-black Gauss-Seidel (1 iteration):

red:

$$\begin{aligned} T_1 = T_7 &= \frac{1}{2}\left(0 - \frac{1}{4}\right) = -\frac{1}{8}, \\ T_3 = T_5 &= \frac{1}{2}\left(-\frac{1}{4} - \frac{1}{2}\right) = -\frac{3}{8}. \end{aligned}$$

black:

$$\begin{aligned} T_2 = T_6 &= \frac{1}{2}\left(-\frac{1}{8} - \frac{3}{8}\right) = -\frac{1}{4}, \\ T_4 &= \frac{1}{2}\left(-\frac{3}{8} - \frac{3}{8}\right) = -\frac{3}{8}. \end{aligned}$$