Basic Equations of Fluid Dynamics

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Focus areas

- Euler and Lagrange descriptions of fluid flows
- The continuity equation (CE)
- The Navier-Stokes (N.-S.) equation
Part 1: Kinematics of fluid flows
Mathematical description of a flow

- The material fluid is considered to be:
  - a continuous substance having a positive volume \((V > 0)\)
  - distributed over a domain \(\Omega\) of the \(d = 3\) Euclidean space

- **A fluid flow:** is represented by a one-parameter family of mappings of a 3d domain \(\Omega\), filled with fluid, into itself, \(\Omega_t \rightarrow \Omega_{t+\tau}\)

**Properties of the mappings:**

- smooth: twice continuously differentiable in all variables, \(C^2\);
  usually mappings are assumed thrice continuously differentiable everywhere, with the exception of some singular points, curves or surfaces where a special analysis is required;

- bijective (one-to-one)

Geometrically: a diffeomorphism (invertible, differentiable), \(\Omega_t \leftrightarrow \Omega_{t+\tau}\)
The fluid flow transform

- The real parameter $t$ designating the mapping of fluid domains is identified with the time, $-\infty < t < +\infty$
- Analytical and numerical description of the fluid motion:
  1) To introduce a fixed rectangular coordinate system $\{x^i\}$
  2) A fluid particle – an infinitesimal element of a fluid distribution
  3) Each coordinate triple $\{x^i\}$ denotes a position of a fluid particle

Without loss of generality, $t = 0$ – an arbitrary initial instant

**Observation:** the particle that was initially in the position $\{\zeta^j\}$ has moved to position $\{x^i\}$

The fluid flow transformation: $x^i = x^i (\zeta^j, t)$

If $\zeta^j$ is fixed while $t$ varies, (*) specifies the path of a fluid particle (a pathline or a particle line): $x^i = x^i (t) = x^i (\zeta^j, t)|_{\zeta^j = \text{const}}$, $x^i(0) = \zeta^i$
Fluid flow – a mapping

• A fluid particle moving with the fluid

\[ M(t) = g_t M \]

\[ g_t : \mathbb{R}^3 \leftrightarrow \mathbb{R}^3 \]

Transformations \( g_t \) make up a one-parameter group of diffeomorphisms, \( g_{t+\tau} = g_t + g_\tau \)
Fluid flow is invertible

• If $t$ is fixed, the transformation of the fluid domain, $\Omega_0 \rightarrow \Omega_t$:
  
  $$ x^i (\xi^j ) = x^i (\xi^j , t) \bigg|_{t=\text{const}} , \quad x^i (0) = \xi^i $$

  is exactly the fluid flow

  $$ x^i (0) = \xi^i \rightarrow x^i (\xi^j , t) $$

Functions $x^i (\xi^j , t)$ are single-valued (1-to-1) and are assumed to be thrice differentiable in all variables. Then

  $$ J = \frac{\partial (x^1 , x^2 , x^3)}{\partial (\xi^1 , \xi^2 , \xi^3 )} = \det \left( \frac{\partial x^i}{\partial \xi^j} \right) \neq 0 $$

Hence, there exists a set of inverse functions: $\xi^j = \xi^j (x^i , t)$ (**)

defining the initial position of a fluid particle, which occupies any position $\{x^i\}$ at time $t$.

Such inverse transform has the same properties as the direct one.
Two equivalent descriptions

• An important consequence of invertibility: **A fluid flow can be equivalently described by a set of initial positions of particles as a function of their positions at later times**

• Thus, there are two equivalent descriptions of the fluid motion:
  - The Eulerian description – in terms of space-time variables \((x^i, t)\)
    Throughout all time, a given set of coordinates \(\{x^i\}\) remains attached to a fixed position. The spatial coordinates \(\{x^i\}\) are called the Euler coordinates; they serve as an identification for a fixed point
  - The Lagrangian description – chronicles the history of each fluid particle uniquely identified by variables \(\{\xi^i\}\)
    Throughout all time, a given set of coordinates \(\{\xi^i\}\) remains attached to an individual particle. They are called the Lagrangian coordinates
Physical interpretation of the Euler and Lagrange descriptions

- Different physical interpretations

In the Euler’s picture, an observer is always located at a given position \{x^i\} at a time \(t\) → observes fluid particles passing by.

In the Lagrange picture, an observer is moving with the fluid particle, which was initially at the position \{\xi^i\} → views changes in the flow co-moving with the observer’s particle.

The Euler description – important for CFD;
The Lagrange description – important for ecological modeling (passive tracers, contamination spread, etc.)
What is a fluid particle?

- Fluid dynamics – the macroscopic discipline studying the continuous media
- The meaning of the word continuous: any “small” fluid element – a fluid particle – is still big enough to contain $N >> 1$ molecules
- Dimensions of a fluid particle: $a << \delta x << L$

$a$ – a typical intermolecular distance
$\delta x$ – a fluid particle diameter
$L$ – a characteristical system dimension
$L$ is an external parameter, e.g. the pipe diameter, the wing length,...
$a \sim N^{-1/3}$, where $N$ is the total number of molecules in the system
In gases, one more parameter is important: the free path length
Field description of fluid flows

The fluid motion is completely determined either by transform (*), Slide 5 – the Euler picture, or by its inverse (**), Slide 7 – the Lagrange picture. The crucial questions:

How to:
- describe the state of motion at a given position in the course of time?
- characterize the fluid motion in terms of some mathematical objects defined at \( \{x^i,t\} \)?

The simplest object for this role is the instant fluid velocity \( u^i \).

\( u \)-functions can be defined in both descriptions, \( u^i(x^j,t) \) and \( u^i(\zeta^j,t) \).

A historical remark: though the descriptions of fluid flows in terms of spatial and material coordinates are attributed, respectively, to Euler and Lagrange, both of them are, in fact, known to be due to Euler (see J.Serrin in Handbuch der Physik, Band 8/1, Springer1959)
Objects for fluid fields

• Consider a function \( f \) characterizing a fluid in the Euler or Lagrange frameworks. Any such quantity (e.g. the temperature) may be viewed as a function of spatial variables, \( f = f(x^i, t) \) and also of material variables, \( f = f(\xi^i, t) \) i.e.

\[
f = f(x^i(\xi^j, t), t) \text{ or } f = f(\xi^j(x^i, t), t)
\]

• Geometrical meaning:
- \( f = f(x^i, t) \) is the value of \( f \) experienced by the fluid particles \text{instantaneously} located at \{\( x^i \}\}
- \( f = f(\xi^i, t) \) is the value of \( f \) felt at time \( t \) by the fluid particle \text{initially} (at \( t = 0 \)) located at \{\( \xi^i \}\}

**Question:** why do we need \( u^i \) and other objects, if we have (*) and (**)? Answer: in most situations the actual motion of fluid particles is not given explicitly i.e. neither \( x^i = x^i(\xi^j, t) \) nor \( \xi^j = \xi^j(x^i, t) \) is known
Change of variables

• The variation of quantity $f$ when the fluid flows – two different time derivatives:

1) $\frac{\partial f}{\partial t} = \frac{\partial f (x^i, t)}{\partial t} \bigg|_{x^i = \text{const}}$ simply a partial derivative

- the rate of change of $f$ with regard to a fixed position $\{x^i\}$

2) $\frac{df}{dt} = \frac{df (\xi^i, t)}{dt} \bigg|_{\xi^i = \text{const}}$ material or convective derivative – the rate of change of $f$ with regard to a moving fluid particle

For $f = x^i$, by definition:

$$u^i (\xi^j, t) = \frac{dx^i}{dt} = \frac{\partial x^i (\xi^j, t)}{\partial t} \bigg|_{\xi^i = \text{const}}$$

- the velocity of a fluid particle
The velocity field

• Usually it is the fluid velocity $u^i(x^j, t)$ that is a measurable quantity. The velocity identifies a fluid particle (in terms of its initial position $\{\xi^i\}$)

• The fluid velocity is a vector field (transformation properties!). Throughout all time, a set of velocity components $\{u^i(x^j, t)\}$ remains attached to a fixed position – the field variables

• In numerics or in experiment: a fluid velocity $u^i (1d, 2d, 3d, 4d)$ for the time interval $\Delta t \rightarrow$ to determine $x^i \approx x^i (\xi^j, t), \Delta x^i \approx u^i \Delta t$
The velocity field (continued)

- Analytically:
  \[ \frac{dx^i}{dt} = u^i(x^j, t), \quad x^i(0) = \xi^i \]  
  (***)

- A dynamical system (the Cauchy problem)
  A unique solution exists on \( U \times T \) for \( u^i \in C^1 \) (Cauchy-Lipschitz-Peano), the solution depends continuously on initial values \( \{\xi^i\} \)

- It means that the fluid motion is characterized completely, i.e. the description in terms of fluid velocity is equivalent to the Euler or Lagrange pictures

- Thus: the state of a fluid in motion is fully represented by the velocity field \( \{u^i(x^j, t)\} \) of a fluid particle at \( \{x^j, t\} \)

- Fundamental importance of the field description: providing the possibility to formulate the fluid dynamics in terms of PDEs.
Other objects

How can one calculate the material derivative of any quantity \( f(x^i, t) \)?

\[
\frac{df}{dt} = \frac{\partial f (\xi^i)}{\partial t} \bigg|_{\xi^i = \text{const}} = \frac{\partial f (\xi^i(x^j, t), t)}{\partial t} \bigg|_{\xi^i = \text{const}} = \frac{\partial f (x^j (\xi^i, t), t)}{\partial t} \bigg|_{\xi^i = \text{const}}
\]

\[
= \frac{\partial f}{\partial t} \bigg|_{\xi^i = \text{const}} + \frac{\partial f}{\partial x^j} \frac{\partial x^j (\xi^i, t)}{\partial t} \bigg|_{\xi^i = \text{const}}
\]

or

\[
\frac{df}{dt} = \frac{\partial f}{\partial t} + u^j \frac{\partial f}{\partial x^j}
\]

1. This formula interconnects the material and spatial derivatives through the velocity field.
2. This formula expresses the change rate of any local parameter \( f(x^i, t) \) with respect to a moving fluid particle located at \( \{x^i, t\} \).

Example: the material derivative – fluid acceleration \( a^i(x^i, t) \)

\[
a^i = \frac{du^i}{dt} = \frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x^j}
\]

Notation: \( D_t u^i := \partial_t u^i + u^j \partial_j u^i = u^i_{,t} + u^j u^i_{,j} \)
A remark on acceleration

- In classical mechanics constructed by Newton, the acceleration plays the central role – in distinction to the Aristotelian picture.
- In fluid dynamics, although it is just a branch of mechanics, acceleration is much less important.
- The reason for it: in most physical and engineering settings acceleration of a fluid particle $a^i$ is uniquely determined by its coordinate $x^i$ and velocity $u^i$.

Therefore, acceleration is usually not considered a state parameter.

So, we might formulate the following field description framework:

The fluid motion can be characterized in terms of “objects” – local parameters of the media - defined at $\{x^i, t\}$.

The simplest object is the fluid velocity $u^i$. 
Streamlines and trajectories

- A streamline - a curve defined for a given $t > 0$, for which the tangent is collinear to the velocity vector $u^i(x^j,t)$:
  \[
  \frac{dx^1}{u^1(x^i,t)} = \frac{dx^2}{u^2(x^i,t)} = \frac{dx^3}{u^3(x^i,t)}
  \]
  Streamlines depend on time $t$ (parameter), for which they are written

- Trajectory of a fluid particle $\{\zeta^i,j,t\}$ (path line, particle line) – a locus of the particle positions for all moments $t > 0$

- For a stationary flow, $u^i(x^j,t) = u^i(x^j)$, streamlines do not depend on $t$ and coincide with particle lines (trajectories)

- The inverse is not true: the flow may be non-stationary, however streamlines can coincide with path lines, for example:
  \[
  u^i = \frac{Q(t)}{4\pi} \frac{x^i}{R^3}, \quad R := (x^i x_i)^{1/2}
  \]
Compressible and incompressible fluids

- Incompressible: when the flow does not result in the change of the fluid volume for all \{x^i, t\}
  \[
  \frac{dJ}{dt} = 0
  \]

- Otherwise - compressible:

  A reminder: the fluid motion is a diffeomorphism, i.e.
  continuously differentiable and invertible

  This is sufficient for \(0 < J < \infty\)

- \(J = dV/dV_0\) – the fluid dilatation

- How to calculate \(dJ/dt\)?
  \[
  \frac{dJ}{dt} = J \frac{\partial \xi^j}{\partial x^i} \frac{dt}{dt} \left( \frac{\partial x^i}{\partial \xi^j} \right) = J \frac{\partial \xi^j}{\partial x^i} \frac{\partial u^i}{\partial \xi^j} = J \partial_i u^i = J \text{div} \mathbf{u}
  \]

\(J\) is the Jacobian of the transform (*),

\[
\begin{align*}
  dV &= JdV_0 \\
  dV &= dx^1 dx^2 dx^3 \\
  dV_0 &= d\xi^1 d\xi^2 d\xi^3
\end{align*}
\]

- the particle volume at \(\{x^i\}\)

- the initial particle volume
Two fundamental statements

- What is the rate of change:

(I) of an infinitesimal volume of a fluid particle?

(II) of any local quantity $f(x^i,t)$ integrated over an arbitrary closed fluid domain (volume) $\Omega(t)$ moving with the fluid?

Answer to (I): the Euler expansion theorem

$$\frac{d}{dt} \ln J = \text{div} \mathbf{u}$$

Answer to (II): the Reynolds transport theorem

A volume integral:

$$F(t) = \int_{\Omega(t)} f(x^i,t) dV, \quad \frac{dF(t)}{dt} = ?$$

The idea: to transform $F(t)$ into an integral over the initial volume $\Omega(0)$, which is fixed

$$\frac{dF}{dt} = \frac{d}{dt} \int_{\Omega(0)} f(x^i(\xi^j,t),t) J dV_0 = \int_{\Omega(0)} \left\{ \left[ \frac{d}{dt} f(x^i(\xi^j,t),t) \right] J + f(x^i(\xi^j,t),t) \frac{dJ}{dt} \right\} dV_0$$
The Reynolds transport theorem

• Using the Euler expansion theorem, we get

\[
\frac{dF(t)}{dt} = \int_{\Omega(0)} \left( \frac{df}{dt} J + f \frac{dJ}{dt} \right) dV_0 = \int_{\Omega(t)} \left( \frac{df}{dt} + f \frac{\partial u^k}{\partial x^k} \right) dV
\]

• Another form of the Reynolds theorem: using the interconnection between the material (Lagrange) and field (Euler) derivatives (see Slide 16), we obtain

\[
\frac{d}{dt} \int_{\Omega(t)} f(x^i, t) dV = \int_{\Omega(t)} \left( \partial_i f + \partial_i (fx^i) \right) dV
\]

• There exists one more form – useful for numerical methods (e.g. for so called mimetic schemes):

\[
\frac{d}{dt} \int_{\Omega(t)} f(x^i, t) dV = \int_{\Omega(t)} \partial_i f dV + \int_{\partial \Omega(t)} fu^k dS_k
\]

Here 2-surface element is a vector normal to \(\partial \Omega(t)\) enclosing \(\Omega(t)\)
Physical interpretation of the Reynolds theorem

- The total rate of change of any quantity $f$ (integrated over the moving volume) = the rate of change of $f$ itself + the net flow of $f$ over the surface enclosing the volume

- A moving (control or material) volume $\Omega(t)$ is a volume of fluid that moves with the flow and permanently consists of the same material particles

- Specific case: $f = \text{const}$ – what is the fluid volume rate of change?

$$V(\Omega, t) := \int_{\Omega(t)} dV; \quad \frac{dV(\Omega, t)}{dt} = \int_{\partial \Omega(t)} \partial_k u^k dV = \int_{\partial \Omega(t)} u^k dS_k$$
Geometry of fluid flows

• A fluid particle moving along its pathline changes its volume, shape, and orientation (e.g. rotates) – these are geometric transforms of a fluid particle (with respect to its initial position)

• From the geometrical viewpoint, such changes can be described as the time evolution of the flow governed by the geodesic equation on the group of volume-preserving diffeomorphisms. So, fluid moves along a geodesic curve

• A dynamical system (***) Slide 15, corresponding to fluid motion, can be visualized as a vector field in the phase space, on which the solution is an integral curve

• Geometry enables us to describe global properties of the family of solution curves filling up the entire phase space

• The geometric approach (originally suggested by V.I. Arnold) is a natural framework to describe the fluid motion
Strain rate and vorticity

- The simplest geometrical object describing spatial variations of a fluid particle is the tensor \( \tau_{ij}(x^k, t) := \partial u_i / \partial x^j \) constructed from the fluid velocity \( u_i = \delta_{ij} u^j \).

- In general, \( \tau_{ij} \) has no symmetry, however it can be decomposed into a symmetric and antisymmetric parts:

\[
\tau_{ij} = \theta_{ij} + \zeta_{ij}, \quad \theta_{ij} = \theta_{ji}, \quad \zeta_{ij} = -\zeta_{ji}
\]

Here the strain rate \( \theta \) and vorticity \( \zeta \) are defined as

\[
\theta_{ij} := \frac{1}{2} \left( \frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i} \right), \quad \zeta_{ij} := \frac{1}{2} \left( \frac{\partial u_i}{\partial x^j} - \frac{\partial u_j}{\partial x^i} \right)
\]

Frequently the vorticity vector is defined as

\[
\zeta^i = \frac{1}{2} \varepsilon^{ijk} \zeta_j \zeta_k, \quad \varepsilon^{ijk} \text{ is the unit antisymmetric pseudotensor of rank 3}
\]

These definitions are used when treating the Navier-Stokes equation.
#### The constitution of fluid kinematics

<table>
<thead>
<tr>
<th>The law of fluid motion</th>
<th>The fluid velocity</th>
<th>IVP for fluid motion</th>
<th>The Euler expansion</th>
<th>The Reynolds transport theorem</th>
<th>The continuity equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x' = x'(\xi, t)$</td>
<td>$\dot{u}(\xi, t) = \partial_x x'(\xi, t)$</td>
<td>$\frac{dx'}{dt} = u(x'(\xi, t), x'(0)) = \xi$</td>
<td>$\int_{\Omega(t)} dJ/dt = \text{div}u$</td>
<td>$\int_{\Omega(t)} \frac{df}{dt} + \text{div}f \text{div}u dV = \int_{\Omega(t)} \text{div}(f dV)$</td>
<td>$\partial_t \rho + \text{div} \rho u = 0$</td>
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Transition to physics $\rho(x', t)$
Control questions on fluid kinematics

1. Can one find the law of motion of a continuous medium, if particle lines are known?

2. Can one find the velocity field, if streamlines are given?

3. Can the particles move with acceleration, if
   a) velocities of all particles are equal?
   b) the velocity does not change with time in each point $x^i \in \Omega$?

4. The density of each individual particle of an incompressible medium is constant. Can the density vary with time in some point $x^i \in \Omega$?
Answers to control questions on fluid kinematics

1. Can one find the law of motion of a continuous medium, if particle lines are known?
   No, since although a curve (trajectory), along which a particle moves, is known for each particle, but the velocity of the motion may be different (is not defined)

2. Can one find the velocity field, if streamlines are given?
   No, in each point the straight line, along which the velocity is directed is known, but the velocity value may be different

3. Can the particles move with acceleration, if:
   a) velocities of all particles are equal? Yes
   b) the velocity does not change with time in each point $x^i \in \Omega$? Yes

4. The density of each individual particle of an incompressible medium is constant. Can the density vary with time in some point $x^i \in \Omega$? Yes
Fluid kinematics – summing it up

- All particles are individualized (classical mechanics!)
- Lagrange coordinates \( \{\zeta^i\} \) are particle identifiers
- Motion of continuous media and accompanying processes are described by physical fields (velocity, pressure, temperature, etc.). In case these fields are considered as functions of \( \zeta^i \), such description is called Lagrangian or material
- Events in the Lagrangian picture occur with individual particles
- The law of motion of a continuous medium: \( x^i = x^i(\zeta^i, t) \)
- Velocity of particles: \( u^i(\zeta^i, t) = \partial_\zeta x^i(\zeta^i, t) \) – the velocity field, i.e. the velocity of particles located in \( \{x^i\} \) at time \( t \)
- Acceleration of particles: \( a^i(\zeta^i, t) = \partial_t u^i(\zeta^i, t) \)
- Fields considered as functions of \( \{x^i, t\} \) – the Euler description
Kinematics – summing up (cont’d)

• Time derivative:
  \[ \frac{df}{dt} = \frac{\partial f}{\partial t} + u^i \frac{\partial f}{\partial x^i} = \left( \partial_t + u^i \partial_i \right) f \]

• Transition from the Lagrangian to the Eulerian description:
  \[ x^i = x^i \left( \xi^j , t \right) \rightarrow \xi^i = \xi^i \left( x^j , t \right) \rightarrow f \left( \xi^i \left( x^j , t \right), t \right) = \varphi \left( x^j , t \right) \]

• Transition from the Eulerian to the Lagrangean description - to solve the Cauchy problem (***)

  \[ \frac{dx^i}{dt} = u^i \left( x^j , t \right) , \quad x^i \big|_{t=0} = \xi^i \]

  The solution (if it could be found):
  \[ \bar{x}^i \left( \xi^j , t \right) \]

  Then for any \( f(x^i , t) \) whose Euler’s description is known:
  \[ f \left( \bar{x}^i \left( \xi^j , t \right), t \right) = \phi \left( \xi^j , t \right) \]
Part2: Dynamics of fluid flows
Transition to fluid dynamics

- No physics so far - no information about the nature or structure of a fluid was required
- How to find the equations governing the fluid behavior?
  1) Mathematical axioms and theorems (mostly kinematics)
  2) Physical hypotheses (structure, interaction law, etc.)
  3) Experimental evidence
  4) Mathematical and computational models
- Transition to dynamics: taking inertia into account (recall the Newton’s law!)
- Fluid specificity: liquid mass is contained within the domain $\Omega(t)$:
  \[ M(t) = \int_{\Omega(t)} \rho \, dV > 0 \quad \text{for any domain } \Omega(t). \]
  Hence $\rho(x^i, t) > 0$, $\{x^i, t\} \in \Omega(t)$
The continuity equation (CE)

- Extremely important!
- The underlying physical principle – conservation of mass (strictly speaking, not a physical conservation law – not associated with any symmetry, a phenomenological principle)
- Leads to a very profound question – what is a mass?

\[
\frac{dM(t)}{dt} = \frac{d}{dt} \int_{\Omega(t)} \rho dV = 0 \quad \text{(in fact along the fluid particle paths)}
\]

- From here – the continuity equation:

\[
\frac{d \rho}{dt} + \rho \frac{\partial u_i}{\partial x^i} = 0
\]

- a necessary and sufficient condition for the mass of any fluid domain to be conserved

Proof is based on the Reynolds transport theorem (see next slide)
The continuity equation - proof

• Sufficiency: to substitute the continuity equation into the Reynolds theorem
  \[
  \frac{dM(t)}{dt} = \int_{\Omega(t)} \left[ \frac{d\rho}{dt} + \rho \frac{\partial u^i}{\partial x^i} \right] dV
  \]

• Necessity: \( \frac{dM(t)}{dt} = 0 \) for any arbitrary \( \Omega(t) \), hence the integrand must be zero.

A more customary form:

\[
\partial_i \rho + \partial_i (\rho u^i) = 0
\]

• The continuity equation, despite its simplicity, brings about many nontrivial results, e.g.:
  - How to find the rate of change of a quantity \( f(x^i, t) \) averaged with the medium density distribution?
  \[
  Q(t) := \int_{\Omega(t)} \rho f dV
  \]
Some consequences of the continuity equation

• The rate of change of the weighted function $f$:

$$\dot{Q}(t) = \frac{d}{dt} \int_{\Omega(t)} \rho(x^i, t) f(x^i, t) dV = ?$$

We have

$$\frac{d}{dt} \int_{\Omega(t)} \rho f dV = \int_{\Omega(t)} \left( \frac{d(\rho f)}{dt} + \rho f \frac{\partial u^k}{\partial x^k} \right) dV =$$

$$= \int_{\Omega(t)} \left( \rho \frac{df}{dt} + \frac{d\rho}{dt} f + \rho f \text{div} u \right) dV =$$

$$= \int_{\Omega(t)} \left( \rho \frac{df}{dt} + f \left( \frac{d\rho}{dt} + \rho \text{div} u \right) \right) dV = \int_{\Omega(t)} \rho \frac{df}{dt} dV$$

The result – a very simple formula:

Due to CE, the differentiation operator ignores the density distribution.
Some more dynamics

- Note: although the medium density is not a kinematic object, the continuity equation may be derived from kinematics.

- Given a fluid density distribution, one can determine a velocity field \( u^i(x^j, t) \) for the fluid motion, which would conserve the mass in each domain \( \Omega(t) \).

- In such formulation, the continuity equation is the first-order PDE with respect to \( u^i(x^j, t) \) – not sufficient to find 3 unknown functions.

Fluid dynamics cannot be described by the continuity equation alone – one needs more physics.
Forces in fluid media

• Physical forces, both internal and external, are acting on fluid particles to set them in motion

• The Cauchy model: two types of forces

✓ internal – contact forces, act on a fluid particle over its surface (e.g. stress, pressure)

✓ external – body forces, act over a medium (e.g. gravity, electromagnetic field)

External forces – described per unit mass by a vector \( F^i(x^j, t, u^k) \) depending on position, time, and state of fluid motion (e.g. magnetic field, relativism, etc.)

• Cauchy: internal forces are more interesting - for any closed surface \( \partial \Omega(t) \), there exists a stress vector distribution \( s^i(x^j, t, n^k) \) depending on the normal \( n^k \)
The Cauchy stress

- The idea: the stress action exercised by the stress vector \( s^i(x^j, t, n^k) \) is equivalent to the action of internal forces \( P \)

\[
s = \lim_{\Delta \sigma \to 0} \frac{\Delta P}{\Delta \sigma}
\]

The Cauchy stress vector

The model: \( s^i \) is a linear function of the normal vector \( n^i(x^j, t) \):

\[
s^i = T^i_j n^j
\]

- So, the stress vector for a surface element with the normal \( n \) can be calculated as \( s_n = sn \)
- The coefficients \( T^i_j \) - the fluid stress tensor
- In general, no symmetry
The Cauchy equation of motion

- An adaptation of the Newton’s law to fluid flows:

$$\rho \frac{du^i}{dt} = \rho F^i + \frac{\partial T^{ij}}{\partial x^j}$$

- System of 1st-order PDE with respect to fluid velocity components
- Includes the density distribution satisfying the continuity equation
- External forces $F^i(x^k,t)$ and the fluid stress tensor $T_{ij}(x^k,t)$ are considered to be known, together with initial and boundary conditions. In this case the solution can be obtained

A system of 1st-order PDE with respect to fluid velocity components

- Includes the density distribution satisfying the continuity equation
- External forces $F^i(x^k,t)$ and the fluid stress tensor $T_{ij}(x^k,t)$ are considered to be known, together with initial and boundary conditions. In this case the solution can be obtained
The general system of equations in fluid dynamics

• If the external forces and the stress tensor components are known, initial and boundary conditions are given, then the fluid velocity $u^i(x^j, t)$ can be obtained from the following system:

\[
\begin{aligned}
\partial_t u + u^j \partial_j u^i &= F^i + \frac{1}{\rho} \partial_j T^{ij} \\
\partial_t \rho + \partial_j (\rho u^j) &= 0
\end{aligned}
\]

- four first-order PDEs over four unknown functions

Once the solution is obtained, the law of motion $x^i = x^i(\xi^j, t)$ can be found from $u^i$ by solving the system of first-order ODEs (***):

\[
\frac{dx^i}{dt} = u^i(x^j, t), \quad x^i(0) = \xi^i
\]
The total acceleration

- Thus, in principle, the problem of fluid motion can be solved (theoretically, in practice this is rather difficult)
- A reduced form of fluid motion equation – analogous to the D’Alembert principle in classical mechanics:

\[ A^i(x^j,t) := \frac{du^i}{dt} - F^i \]

- the total acceleration, a combined response to inertial and external forces

\[ \rho A^i = \partial_j T^{ij} \]

**Physical interpretation:** total acceleration is determined by internal forces (acting on a fluid particle)

- One needs **physical** assumptions regarding the nature of internal forces (acting across fluid surface elements)
- Such assumptions would specify the \( T^{ij} \) tensor components
Specific forms of motion equations

No a priori properties of $T^{ij}(x^k, t)$: no algebraic symmetry (indices), no geometric symmetry ($x'^k = R^k_i x^i$), no physical symmetry ($t \rightarrow -t$)

• An example of a physical assumption – the Boltzmann postulate:

\[ T^{ij} = T^{ji} \]

– an immediate consequence: conservation of angular momentum

Holds not for all fluids: e.g. for polar fluids the asymmetry of $T^{ij}$ may be essential

• Now, we restrict ourselves to fluids without heat transfer: the temperature $T = \text{const}$, the heat flux $q^i(x^i, t) = 0$

Three main fluid equations:
1. The Euler equation
2. The Stokes equation;
3. The Navier-Stokes equation
The Euler equation

- The perfect fluid, $T^{ij} = -p \delta^{ij}$:

$$\frac{\partial}{\partial t} u^i + u^j \frac{\partial}{\partial x^j} u^i = F^i - \frac{1}{\rho} \delta^{ij} \frac{\partial}{\partial x^j} p$$

$p(x^j,t)$ is the fluid pressure – depends on the thermodynamic state

**A control question:** can the pressure be $< 0$?

**Answer:** yes – expansion of a fluid particle (in fact instability)

What is a perfect fluid? (“Nobody‘s perfect!“)

- No dissipation, no internal friction, no internal heat transfer

Heat conductivity and viscosity are neglected – adiabatic motion:

$$ \frac{ds}{dt} = \frac{\partial s}{\partial t} + u^i \frac{\partial s}{\partial x^i} = 0 $$

Combining with the continuity equation:

$$ \frac{\partial (\rho s)}{\partial t} + \frac{\partial}{\partial x^i} (\rho s u^i) = 0 $$

the entropy flux
Another form of the Euler equation

Introducing $h$ – enthalpy, $dh = Tds + Vdp$:

$$\frac{\partial u^i}{\partial t} + u^i \frac{\partial u^j}{\partial x^j} = -\frac{\partial h}{\partial x^i}$$

(for $ds = 0$, $(1/\rho)\partial \rho/\partial x^i = \partial h/\partial x^i$)

Using the formula:

$$(u \nabla)u = \frac{1}{2} \nabla (u^2) - [u \ curl u]$$

we can write

$$\frac{\partial u}{\partial t} - [u \ curl u] = -\nabla (h + \frac{u^2}{2})$$

or (applying curl):

$$\frac{\partial}{\partial t} curl u = curl [u \ curl u]$$

- only the velocity is present

Boundary condition: $u_n = 0$ (fixed boundary)

In general, 5 variables: $u^i, \rho, p; \ \rho = \rho(p,T)$ – the state equation
The model:

\[ T^{ij} = (-p + \alpha)\delta^{ij} + \beta\theta^{ij} + \gamma\delta_{kl}\theta^{ik}\theta^{jl} \]

The viscosities:

\( \alpha(x^i, t), \beta(x^i, t), \gamma(x^i, t) \) – scalar functions, in general depend on the thermodynamical state of the fluid

\[
\rho \left( \partial_t u^i + u^j \partial_j u^i \right) = \rho F^i - \delta^{ij} \partial_j \rho + \\
+ \partial_j (\beta \theta^{ij}) + \delta_{kl} \partial_j (\gamma \theta^{ik} \theta^{jl})
\]

- a very complex expression.

The Stokes equation is used to model non-elastic fluids
The Navier-Stokes equation

- The model:

\[
T^{ij} = (\lambda \theta - p)\delta^{ij} + 2\mu \theta^{ij} \\
\rho \left( \partial_t u^i + u^j \partial_j u^i \right) = \rho F^i - \delta^{ij} \partial_j \rho + \delta^{ij} \partial_j (\lambda \theta) + 2\partial_j (\mu \theta^{ij})
\]

Here \( \lambda(x^i, t) \), \( \mu(x^i, t) \) are scalar functions - the first and the second viscosities. For \( \lambda = \text{const} \) and \( \mu = \text{const} \):

\[
\frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x^j} = F^i - \frac{1}{\rho} \delta^{ij} \frac{\partial p}{\partial x^j} + \frac{\lambda + \mu}{\rho} \delta^{ij} \frac{\partial^2 u^k}{\partial x^i \partial x^k} + \mu \delta^{kl} \frac{\partial^2 u^i}{\partial x^k \partial x^l}
\]

For \( \rho = \text{const} \) (incompressible), \( \text{div } u = 0 \). Then

\[
\frac{\partial u}{\partial t} + (u \nabla) u = F - \frac{\nabla p}{\rho} + \nu \Delta u \quad \left( \nu = \frac{\mu}{\rho} \right)
\]

- the main equation of the classical fluid dynamics
Another form of the Navier-Stokes equation

- Similarly to the Euler equation – an alternative form
- Incompressible fluid $\text{div } \mathbf{u} = 0 \rightarrow$ the velocity field is rotational or transversal, $\mathbf{u} = \mathbf{u}_\perp$. Applying $\text{curl}$ (or $\text{rot}$) to the N.-S. equation:

\[
\frac{\partial}{\partial t} \text{curl} \mathbf{u} = \text{curl}[\mathbf{u} \cdot \text{curl} \mathbf{u}] + \nu \Delta \text{curl} \mathbf{u} + \text{curl} \mathbf{F}
\]

or

\[
\frac{\partial}{\partial t} \text{curl} \mathbf{u} + (\mathbf{u} \nabla) \text{curl} \mathbf{u} - (\text{curl} \mathbf{u} \nabla) \mathbf{u} = \nu \Delta \text{curl} \mathbf{u} + \text{curl} \mathbf{F}
\]

This is a closed equation with respect to $\mathbf{u}$ – no pressure involved

On the contrary, equation for the pressure:

or, transforming,

\[
\Delta p = -\rho \frac{\partial^2 (u^i u^j)}{\partial x^i \partial x^j}
\]

\[
\Delta p = -\rho \frac{\partial u^i}{\partial x^j} \frac{\partial u^j}{\partial x^i} = -\rho \tau^i_j \tau^j_i, \tau^i_j := \partial_j u^i
\]
Derivation of the pressure equation

- Applying $\text{div}$ to the Navier-Stokes equation:
  \[
  \frac{\partial}{\partial t}\left(\frac{\partial u^i}{\partial x^i}\right) + \frac{\partial}{\partial x^j}\left(u^j \frac{\partial}{\partial x^j}\right) u^i = -\frac{\Delta p}{\rho} + \frac{\partial}{\partial x^i}\left(\nu \frac{\partial^2}{\partial x^j \partial x^j}\right) u^i + \frac{\partial F^i}{\partial x^i}
  \]
  Assuming $\text{div} \; \mathbf{F} = 0$ (no charges). The second term in LHS:
  \[
  \frac{\partial}{\partial x^i}\left(u^j \frac{\partial}{\partial x^j}\right) u^i = \frac{\partial u^j}{\partial x^i} \frac{\partial u^i}{\partial x^j} + u^j \frac{\partial^2 u^i}{\partial x^j \partial x^j} = \frac{\partial u^j}{\partial x^i} \frac{\partial u^i}{\partial x^j} + u^j \frac{\partial}{\partial x^j} \frac{\partial u^i}{\partial x^i} = \frac{\partial u^i}{\partial x^i} \frac{\partial u^i}{\partial x^i}
  \]
  The second term in the RHS:
  \[
  \frac{\partial}{\partial x^i}\left(\nu \frac{\partial^2 u^i}{\partial x^j \partial x^j}\right) = \nu \frac{\partial^2}{\partial x^j \partial x^j} \frac{\partial u^i}{\partial x^i} = 0
  \]
  Thus:
  \[
  \Delta p = -\rho \frac{\partial u^i}{\partial x^j} \frac{\partial u^j}{\partial x^j} = -\rho \tau_i^j \tau_i^j = \partial_j u^i
  \]
  since \[
  \frac{\partial^2}{\partial x^i \partial x^j} \left( \frac{\partial u^i}{\partial x^j} + u^j \frac{\partial u^i}{\partial x^j} \right) = \frac{\partial u^j}{\partial x^i} \frac{\partial u^i}{\partial x^j}
  \]
  or, in an equivalent form,
  \[
  \Delta p = -\rho \frac{\partial^2 (u^i u^j)}{\partial x^i \partial x^j}
  \]
A closed system of equations of classical fluid dynamics

- A closed system of equations consists of:
  - The Navier-Stokes (N.-S.) equation
  - The continuity equation
  - The thermodynamic state equation \( \rho = \rho(p, T) \)

5 equations for unknown functions: \( u^i(x^j, t), p(x^j, t), \rho(x^j, t) \)

In principle, the system is closed, provided the external forces \( F^i(x^j, t) \) and the viscosity coefficients \( \lambda, \mu \) are given

- Initial and boundary conditions must be imposed to select an appropriate solution

- Although considerable difficulties do exist in solving the N.-S. equation, it has been successfully applied to described various regimes of fluid motion (including turbulence – to some extent)
Elementary model of the viscosity

- Many practically important problems cannot be solved without viscosity being taken into account, e.g.
  - motion of bodies in fluids
  - fluid flow through channels, tubes, etc.
  - flows past an obstacle

Force applied to a unit volume:

\[ F^x(y + dy) = \mu dxdz \frac{du^x}{dy} \bigg|_{y+dy} \]

\[ F^x(y) = -\mu dxdz \frac{du^x}{dy} \bigg|_y \]

\[ f^x = \frac{F^x(y + dy) - F^x(y)}{dxdydz} = \mu \frac{\partial^2 u^x}{\partial y^2} \]

To N.-S. equation
Newtonian fluids

- When a solid is slightly deformed so that it is strained, a restoring force opposing the strain is observed; for small strain it is proportional to the strain (the Hooke’s law)
- Fluids also resist strains, but for a fluid it is not only the strain magnitude that is important, but the strain rate as well
- Fluids responding to a sheer stress independent of its rate are known as Newtonian fluids. More generally, in a Newtonian fluid the sheer stress is proportional to the velocity gradient (see Slide 49), i.e. components of the stress tensor are linear functions of $\tau^{ij} = \partial u^i / \partial x_j$
- For Newtonian fluids, the viscosity is constant (does not depend on how fast the upper plate is sliding past the bottom)
- For non-Newtonian fluids, viscosity depends on relative velocity. Such fluids (studied in rheology) may have no defined viscosity
The physical meaning of viscosity

- Viscosity is a macroscopic manifestation of intermolecular interactions. Viscosity always leads to energy dissipation.
- All phenomena involving viscosity are irreversible.
- We can calculate the energy dissipation rate in an incompressible viscous fluid – no work is spent on compression (or rarefaction) of fluid particles. The rate of change of kinetic energy inside the control volume $\Omega$:

$$\dot{E}(\Omega) = \int_{\Omega} \rho u_i \frac{\partial u^i}{\partial t} dV = \int_{\Omega} u_i \left( -\rho u^k \frac{\partial u^i}{\partial x^k} - \frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u^i}{\partial x^k \partial x^k} \right) dV$$

Since

$$u_i \frac{\partial^2 u^i}{\partial x^k \partial x_k} = \frac{\partial}{\partial x^k} \left( u_i \frac{\partial u^i}{\partial x^k} \right) - \left( \frac{\partial u^i}{\partial x^k} \right)^2, \quad \rho u_i \left( u^k \frac{\partial u^i}{\partial x^k} + \frac{1}{\rho} \frac{\partial p}{\partial x_i} \right) = \frac{\partial}{\partial x_i} \left[ \rho u_i \left( \frac{u^2}{2} + \frac{p}{\rho} \right) \right]$$

we get

$$\dot{E}(\Omega) = -\int_{\partial \Omega} d\sigma^k \left[ \rho u_k \left( \frac{u^2}{2} + \frac{p}{\rho} \right) - \mu u_i \frac{\partial u^i}{\partial x^k} \right] - \mu \int_{\Omega} \left( \frac{\partial u^i}{\partial x^k} \right)^2 dV$$
The physical meaning of viscosity (cont’d)

• The surface integral in $\dot{E}(\Omega)$ describes the energy change due to the fluid inflow/outflow (the term $\rho u^k u^2/2$) and with the work produced by surface stresses

• The dissipated power is given by the volume integral, which is always positive

• The viscous dissipation rate per unit volume is $\dot{Q} = \mu \left( \frac{\partial u^i}{\partial x^k} \right)^2 = \mu \tau^{ik} \tau_{ik}$
i.e. proportional to the dynamic viscosity

• The viscous coefficient $\mu \geq 0$ (this is the consequence of the second law of thermodynamics)

When can one neglect viscosity?

• The viscous force $\mu \Delta u \sim \mu u/L^2$ must be $<< \nabla p \sim p/L \sim \rho u^2/L$ or $\mu << \rho uL$ (here $L$ is the system’s characteristic size, see Slide 10)
This criterion can be written as $1/Re << 1$, $Re := \rho uL/\mu = uL/\nu$
The Reynolds number

- In 1883 Reynolds made the systematic study of the water flow through a circular tube and found that the flow pattern changes from a “laminar“ to a “turbulent“ state at a certain value of $uL/\nu$
- Later, this quantity was called after him the Reynolds number, $Re$

Using the Reynolds number, one can write the N.-S. equation in dimensionless form:

$$\frac{Du^i}{Dt} = -\delta^{ij} \partial_j p + Re^{-1} \partial^j \partial_j u^i + F^i$$

($D := \partial_t + u^j \partial_j$, see Slide 16 – in fact a covariant derivative)

- Physically, $Re$ is a measure of the ratio between inertial and viscous forces in the flow; for $Re>>1$ inertia prevails, for $Re<<1$ friction (viscosity) prevails
- Both forces are balanced by the pressure gradient
- Simple steady flows at small values of $Re$ vs. complicated non-stationary flows at large values of $Re$
Flow past an obstacle

• The transition from a laminar to a turbulent mode can be observed in the flow past an obstacle, the simplest obstacle being a sphere.

\[
\begin{align*}
\text{Re} &\leq 1 \\
\text{Re} &\approx 10 \\
\text{Re} &\approx 10^2 - 10^5 \\
\text{Re} &> 10^5
\end{align*}
\]
Description of flow patterns

- \( \text{Re} \lesssim 1 \): The flow is stationary and passes regularly around the sphere (without separation)
- \( 1 \lesssim \text{Re} \lesssim 10 \): The flow is still stationary but separates at the back side of the sphere. A ring vortex is formed, which increases with \( \text{Re} \)
- \( 10 \lesssim \text{Re} \lesssim 10^2 \): The ring vortex deforms into a helical vortex, which rotates about the flow axis. The flow becomes non-stationary, but periodic (pulsations)
- \( 10^2 \lesssim \text{Re} \lesssim 10^5 \): The helical vortex breaks up and is replaced by a turbulent wake. The flow is totally aperiodic in the wake
- \( \text{Re} > 10^5 \): Not only the wake, but the entire flow (including the boundary layer) becomes turbulent. The flow is completely random everywhere. The wake is thinner and the drag reduced
Mathematical status of the N.-S. equation

• No existence theorem is known as yet – for a smooth and physically reasonable solution to a 3d Navier-Stokes equation
• Two major sources of mathematical difficulties:
  1) a nonlinear (quasilinear) system of equation – due to the convection (inertial) term $u^j \partial_j u^i$
     This nonlinearity is of kinematic (geometrical) origin, arising from mathematical derivation (the chain rule), not from physical considerations. A unique case in physics
  2) viscosity terms introduce the second-order operator into the first-order equation – leads to singular perturbations

The question: would the solutions to the N.-S. equations in general converge to the solutions of the Euler equations in a domain $\Omega$ with boundary $\partial \Omega$, as the viscosity tends to zero?
Physical status of the N.-S. equation

- A phenomenological macroscopic equation – to describe the dynamics of fluids viewed as continuous media
- The fluid flow at the microscopic level is treated in non-equilibrium statistical mechanics and physical kinetics
- Physics of fluids can be reconstructed from the first principles only for very special cases (e.g. dilute gases)
- The N.-S. equation, the Euler equation, etc. can be obtained from the Boltzmann equation (the moments of Boltzmann equation)
- The Boltzmann equation is also a special case of a more general dissipative kinetic equation, the latter being in its turn a specific case of a more general equation (the Liouville equation), etc.
- So, the Navier-Stokes equation is accepted only as a reasonable mathematical model for fluid flows
Phenomenology of fluid motion

- From the physical viewpoint, the fluid in motion is a macroscopic non-equilibrium system
- When physicists are studying such systems, they apply two distinct approaches: phenomenological and microscopic
- In the phenomenological method, the problem is reduced to establishing the relationship between macroscopic parameters, without referring to atomic or molecular considerations
- The concept of continuous medium is an idealization. When the atomic structure is taken into account, fluctuations become essential. One can consider their influence only within the framework of kinetic theory.
- The meaning of stress tensor, viscosity, heat conductivity, etc., as well as the applicability of the N.-S. equation can be also elucidated only within the framework of kinetic theory
Examples of using the distribution function

• The kinetic equation is with respect to distribution (one-particle) function, PDF: $f(r, v, t) = f(x^i, v^i, t)$

• Definition of fluid dynamics quantities in terms of PDF:

$$\rho(x^i, t) = \int f(x^i, v^i, t) d^3v$$

• For an incompressible fluid:

$$\int f(x^i, v^i - u^i, t) d^3v = \text{const}$$

where

$$u^i = \int v^i f(x^i, v^i, t) d^3v$$

the average flow velocity (the main quantity in the fluid dynamics)

The flow temperature:

$$T = \frac{m}{2} \int (v^i - u^i)^2 f(x^i, v^i - u^i, t) d^3v$$

We assumed it to be constant, thus disregarding the heat transfer
What else?

Physical fluid dynamics:
- Heat transfer
- Thermodynamics of continuous media
- Surface phenomena and free boundaries
- Dimensional analysis, scaling, and self-similarity
- Boundary layers
- Compressible fluids and gas dynamics
- Turbulence

Computational fluid dynamics (CFD) and numerical modeling