Godunov/Roe Fluxes

Oliver Meister

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Integral form of Conservation Law

Start point: Integral form of a general conservation law in 1D:
For any $x_1, x_2, t_1, t_2 \in \mathbb{R}$ the quantity q is conserved if:

$$\int_{x_1}^{x_2} q(x, t_2) - q(x, t_1)\, dx + \int_{t_1}^{t_2} f(q(x_2, t)) - f(q(x_1, t))dt = 0 \quad (1)$$

where:

- $x, t \in \mathbb{R}$: space and time variables
- $q : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^d$: conserved quantity vector (function over space and time)
- $f : \mathbb{R}^d \to \mathbb{R}^d$: flux function

FV discretization

Finite Volumes: subdivision of domain into *grid cells*, each denoted by a range $C_i := (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$.
Similarly, we choose a finite set of time steps $(t_n)_{n=1...N}$ This gives the following update rule:

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} q(x, t_{n+1}) - q(x, t_n) dx = \int_{t_n}^{t_{n+1}} f(q(x_{i-\frac{1}{2}}, t)) - f(q(x_{i+\frac{1}{2}}, t)) dt \tag{2}$$

Note: No approximation has been performed. So far we have an exact solver.

FV discretization

Discretization:

Average q over space by $Q_i^n := \frac{1}{\Delta x} \int\limits_{C_i} q(x, t_n) dx$.

Average $f(q)$ over time by $F_{i+\frac{1}{2}}^n := \frac{1}{\Delta t} \int\limits_{t_n}^{t_{n+1}} f(q(x_{i+\frac{1}{2}}, t)) dt$.

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$$\Rightarrow Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n)$$

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Problem: Assume Q_i^n is known, how do we get $F_{i-\frac{1}{2}}^n$, $F_{i+\frac{1}{2}}^n$ in order to compute Q_i^{n+1}?

An unstable method

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$$F_{i+\frac{1}{2}}^n = \mathcal{F}(Q_i^n, Q_{i+1}^n)$$

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First attempt:

$$\mathcal{F}(Q_i^n, Q_{i+1}^n) := \frac{1}{2}(f(Q_i^n) + f(Q_{i+1}^n))$$

Resulting method:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x}(f(Q_{i+1}^n) - f(Q_{i-1}^n))$$

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$\Rightarrow$ Central difference term, unstable.

Lax-Friedrichs Method

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$\Rightarrow$ Lax-Friedrichs Method, stable but diffusive.

Upwind Method

Idea: Use Riemann problems for scalar conservation laws in order to approximate fluxes

- Riemann problem: constant function with a single discontinuity at $x = 0$ in an infinite domain
- Finite Volumes: piecewise constant function with discontinuities between cell interfaces
- Find a single shock solution on the cell interfaces $\rightarrow$ solution q on the interface is either left state q_l or right state q_r. Then $f(q) = f(q_l)$ or $f(q_r)$.

Upwind Method

Result:

$$Q_i^{n+1} = \begin{cases} Q_i^n - \frac{\Delta t}{\Delta x}(f(Q_i^n) - f(Q_{i-1}^n)) & f'(Q_i^n) \geq 0 \\ Q_i^n - \frac{\Delta t}{\Delta x}(f(Q_{i+1}^n) - f(Q_i^n)) & f'(Q_i^n) < 0 \end{cases}$$

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$\Rightarrow$ Godunov's method, etc.

Godunov's method

A step further:

- Find the solution to the general Riemann problem (see last talk).
- Evaluate the solution at the cell interface. Due to the self-similarity of the Riemann solution, the result q° is constant in time.
- $\Rightarrow$ The numerical flux $\frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f(q(x,t))dt$ reduces to the simple formula $f(q^\circ)$

Godunov's method with Roe linearization

General Riemann problem is far too expensive, hence we try to approximate the solution by linearization.

Goal: For each cell interface, find matrix $\hat{A} \in \mathbb{R}^{d\times d}$ such that the system

$$\hat{q}_t + \hat{A}_{i-\frac{1}{2}}\hat{q}_x = 0$$

approximates the original system

$$q_t + f'(q)q_x = 0$$

by $\hat{A}_{i-\frac{1}{2}} \rightarrow f'(\tilde{q})$ as $Q_{i-1}, Q_i \rightarrow \tilde{q}$.

Godunov's method with Roe linearization

Assumption: A single shock wave with speed s connects Q_i and Q_{i+1}, so that

$$f(Q_i) - f(Q_{i-1}) = s \cdot (Q_i - Q_{i-1})$$

Our approximation must meet

$$\hat{A}_{i-\frac{1}{2}}(Q_i - Q_{i+1}) = s \cdot (Q_i - Q_{i-1}) = f(Q_i) - f(Q_{i-1})$$

Godunov's method with Roe linearization

Set:

$$q(\xi) := Q_{i-1} + \xi \cdot (Q_i - Q_{i-1})$$

Then

$$f(Q_i) - f(Q_{i-1}) = \int_0^1 \frac{df(q(\xi))}{d\xi} d\xi =$$

$$\int_0^1 f'(q(\xi))q'(\xi)d\xi = \left[\int_0^1 f'(q(\xi))d\xi\right](Q_i - Q_{i-1})$$

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$$\int_0^1 f'(q(\xi)) q'(\xi) d\xi = \left[\int_0^1 f'(q(\xi)) d\xi \right] (Q_i - Q_{i-1})$$

So a suitable choice for $\hat{A}$ is $\int_0^1 f'(q(\xi)) d\xi$.

Godunov's method with Roe linearization

Usually, instead of integrating over ξ a transformation $z(\xi)$ is used to integrate a path on, because the resulting system might not be hyperbolic any more.

$$f(Q_i) - f(Q_{i-1}) = \left[\int_0^1 f'(q(z(\xi)))d\xi \right] (Z_i - Z_{i-1})$$

where $Z_i = z(Q_i), Z_{i-1} = z(Q_{i-1})$ and

$$Q_i - Q_{i-1} = \left[\int_0^1 \frac{dq(z(\xi)))}{dz} d\xi \right] (Z_i - Z_{i-1})$$

Roe solver for the 1D Shallow Water Equations

Nonlinear system of equations (ignoring source terms):

$$\begin{bmatrix} h \\ hu \end{bmatrix}_t + \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix}_x = 0$$

They can be described in quasilinear form:

$$\begin{bmatrix} h \\ hu \end{bmatrix}_t + \begin{bmatrix} 0 & 1 \\ -(u)^2 + gh & 2u \end{bmatrix} \begin{bmatrix} h \\ hu \end{bmatrix}_x = 0$$

Roe solver for the 1D Shallow Water Equations

Choose as parameter vector $z := \frac{1}{\sqrt{h}} q$. Leaving out intermediate steps we obtain:

$$\hat{A} = \begin{bmatrix} 0 & 1 \\ -\hat{u}^2 + g\bar{h} & 2\hat{u} \end{bmatrix}$$

as our Roe matrix with the arithmetic average $\bar{h}$ and the *Roe average* $\hat{u}$. Applying Godunov's method for the time step, we have a full numerical scheme now.