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Algorithms for Uncertainty Quantification

Tutorial 7: Polynomial chaos approximation 2: stochastic Galerkin

In this worksheet, we focus on the generalized polynomial chaos approximation and the stochastic Galerkin projection.

Stochastic Galerkin projection

In the previous tutorial we discussed the pseudo-spectral approach to estimating the coefficients of the generalized polynomial chaos approximation . With this approach, the aforementioned coefficients are computed via quadrature, chosen with respect to the input probability distribution.
In the lecture, we saw an alternative method to compute the coefficients of the polynomial chaos expansion, the stochastic Galerkin method. In a nutshell, instead of using quadrature nodes and weights, we replace the uncertain quantities with their polynomial chaos expansion approximation and ensure that the resulting residual is projected on the basis consisting of the orthogonal basis. Therefore, the stochastic Galerkin method requires explicit access to the mathematical model of the problem under consideration. Since the stochastic Galerkin approach relies on replacing the uncertain quantities in the model's equation via their polynomial chaos expansion approximation, the first step is to write the uncertain inputs in a polynomial chaos expansion format.

Assignment 1

Consider $U \sim \mathcal{U}(0, 2)$ and $N \sim \mathcal{N}(10, 1)$. Derive the polynomial chaos expansion of U and N. *Hint: write U and N in terms of standard random variables; in this context, the standard uniform distribution is $\mathcal{U}(-1, 1)$.*

Assume that the underlying model is of the form $f(x, \omega)$, where x denotes the deterministic and ω the stochastic inputs. After we have the polynomial chaos expansion of ω, we write the N^{th} order polynomial chaos expansion of $f(x, \omega)$ as

$$f(x,\omega) \approx \sum_{i=0}^{N-1} \hat{f}_i(x)\ \phi_i(\omega), \tag{1}$$

and we plug both the expansion of ω and Eq. (1) into the underlying model equation. Afterwards, we perform the Galerkin projection step (in other words, we take advantage of the orthogonality of the polynomial basis) and obtain a system of (coupled) equations whose unknowns are $\hat{f}_i(x), \quad i = 0, \dots, N-1$.

Solution

The first key step is to write the random variable in terms of standard random variables (*see Tutorial 2*).

Therefore, in case of a uniform random variable $U_g \sim \mathcal{U}(a, b)$, we need to write U_g in terms of $U_s \sim \mathcal{U}(-1, 1)$. That means, $U_g = \underbrace{\frac{b+a}{2}}_{c_1} + \underbrace{\frac{b-a}{2}}_{c_2} U_s = c_1 1 + c_2 U_s$. The next key point is that in every series of orthogonal polynomials, $\Phi_0(x) \equiv 1, \Phi_1(x) = x$. Thus,

$$U_g = c_1 1 + c_2 U_s = c_1 \Phi_0(u_s) + c_2 \Phi_1(u_s) = \sum_{i=0}^{1} \hat{u}_i \Phi_i(u_s), \quad \hat{u}_0 = c_1, \hat{u}_1 = c_2.$$

When $U_g = U \sim \mathcal{U}(0, 2), U_g = 1 + U_s = \Phi_0(u_s) + \Phi_1(u_s)$.

Applying the same thinking for $N_g \sim \mathcal{N}(\mu, \sigma^2), N_s \sim \mathcal{N}(0, 1)$, we have

$$N_g = \mu 1 + \sigma N_s = \mu \Phi_0(u_s) + \sigma \Phi_1(u_s) = \sum_{i=0}^{1} \hat{u}_i \Phi_i(u_s), \quad \hat{u}_0 = \mu, \hat{u}_1 = \sigma.$$

When $N_g = N \sim \mathcal{N}(10, 1), N_g = 10 + N_s = 10\Phi_0(n_s) + \Phi_1(n_s)$.

Assignment 2

Consider the model problem, the linear damped oscillator

$$\begin{cases} \frac{d^2y}{dt^2}(t) + c\frac{dy}{dt}(t) + ky(t) = f\cos(\omega t) \\ y(0) = y_0 \\ \frac{dy}{dt}(0) = y_1. \end{cases} \tag{2}$$

Assume that c, f, w, y_0, y_1 are known but $k \sim \mathcal{U}(0, 2)$. Approximate $y((c, f, w, y_0, y_1, t), k)$ via a degree N polynomial chaos expansion. Assume, for simplicity, that the orthogonal basis functions are normalized, i.e. they are orthonormal. Employ the stochastic Galerkin approach to derive the system of equations needed to compute the polynomial chaos expansion's coefficients.

Solution

The solution is pretty much given in the lecture slides. For brevity, we concentrate only on the derivation of the modified ODE system, omitting the derivation of the modified initial conditions. Using the results from Assignment 1, we have

$$k = 1 + U_s = 1 + \Phi_1(u_s). \tag{3}$$

Furthermore, we need also write the solution in terms of a polynomial chaos expansion

$$y(t, u_s) = \sum_{i=0}^{N-1} \hat{y}_i(t)\Phi_i(u_s) \tag{4}$$

Plugging Eq. (3) and (4) into Eq. (2), we have

$$\frac{d^2y}{dt^2}(t, u_s) + c\frac{dy}{dt}(t, u_s) + ky(t, u_s) = f\cos(\omega t)$$

$$\Leftrightarrow$$

$$\frac{d^2\left(\sum_{i=0}^{N-1} \hat{y}_i(t)\Phi_i(u_s)\right)}{dt^2} + c\frac{d\left(\sum_{i=0}^{N-1} \hat{y}_i(t)\Phi_i(u_s)\right)}{dt}(t, u_s) + (1 + \Phi_1(u_s))(\sum_{i=0}^{N-1} \hat{y}_i(t)\Phi_i(u_s)) = f\cos(\omega t)$$

$$\Leftrightarrow$$

$$\sum_{i=0}^{N-1} \frac{d^2\hat{y}_i(t)}{dt^2}\Phi_i(u_s) + c\sum_{i=0}^{N-1} \frac{d\hat{y}_i(t)}{dt}\Phi_i(u_s) + \sum_{i=0}^{N-1} \big(1 + \Phi_1(u_s)\big)\hat{y}_i(t)\Phi_i(u_s) = f\cos(\omega t)$$

$$\Leftrightarrow$$

$$\sum_{i=0}^{N-1} \frac{d^2\hat{y}_i(t)}{dt^2}\Phi_i(u_s) + c\sum_{i=0}^{N-1} \frac{d\hat{y}_i(t)}{dt}\Phi_i(u_s) + \sum_{i=0}^{N-1} \hat{y}_i(t)\Phi_i(u_s) + \sum_{i=0}^{N-1} \hat{y}_i(t)\Phi_1(u_s)\Phi_i(u_s) = f\cos(\omega t)$$

In the next step, we perform the Galerkin projection, i.e. we multiply both sides of the above equation with $\Phi_j(u_s)$ and then take the expectation, i.e. inner product, computed with respected to the given probability density function. To simplify notation, we omit the argument u_s in the orthogonal polynomials Φ_1, Φ_i, Φ_j.

$$\sum_{i=0}^{N-1} \frac{d^2\hat{y}_i(t)}{dt^2}\langle\Phi_i, \Phi_j\rangle + c\sum_{i=0}^{N-1} \frac{d\hat{y}_i(t)}{dt}\langle\Phi_i, \Phi_j\rangle + \sum_{i=0}^{N-1} \hat{y}_i(t)\langle\Phi_i, \Phi_j\rangle + \sum_{i=0}^{N-1} \hat{y}_i(t)\langle\Phi_1\Phi_i, \Phi_j\rangle = \langle f\cos(\omega t), \Phi_j\rangle$$

Since the orthogonal polynomials are orthonormal, denoting $\langle \Phi_1 \Phi_i, \Phi_j \rangle = \Phi_{1ij}$, the above equation becomes

$$\frac{d^2 \hat{y}_i(t)}{dt^2} + c\frac{d\hat{y}_i(t)}{dt} + \hat{y}_i(t) + \sum_{i=0}^{N-1} \hat{y}_i(t)\Phi_{1ij} = f\cos(\omega t)\delta_{0i}$$