General Instructions

Material:

You may only use one hand-written sheet of paper (size A4, on both pages).

Any other material including electronic devices of any kind is forbidden.

Use only the exam paper that was handed out to solve the exercises. In case the space on a page is not enough, mark that you continue with your solution and use the reverse side of the preceding page. For additional notes and sketches, you can obtain additional exam sheets.

Do not use pencil, or red or green ink.

General hint:

Often, exercises b), c), etc. can be solved without the results from the previous exercise a); if you are stuck with exercise a), then don't immediately skip exercises b), c), etc.

Working time:

75 minutes + 5 minutes reading time.

Please switch off your cell phones!

Good luck!

1	2	3	4	5	$\sum$
/11	/11	/10	/6	/4	/≈ 42

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1 Sampling Methods (≈ 3 + 2 + 6 = 11 points)

In this problem set, the focus is on sampling methods.

(a) You are given a set of five samples $X = \{1.80, 1.66, 1.82, 1.72, 1.60\}$ representing five persons' heights in meters. Moreover, you are told that the mean value is 1.72 and the variance is 0.00688. Are these estimates coming from biased or unbiased estimators? Show your calculations.

(b) Evaluate $\int_{-\infty}^{\infty} (x + x^2) \exp{(-x^2/2)} dx$

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(c) Briefly explain what standard and quasi-Monte Carlo methods are, including two commonalities and two differences

2 Non-intrusive Approaches (≈ 2 + 2 + 3 + 4 = 11 points)

(a) Which class of orthogonal polynomials is related to uniform distributions in the context of UQ? And which to normal distributions?

(b) Explain the curse of dimensionality. What approach can be used to circumvent or delay it?

(c) Consider the *Ishigami* function $f : [-\pi, \pi]^3 \to \mathbb{R}$

$$f(\boldsymbol{x}) = \sin(x_1) + a\sin^2(x_2) + bx_3^4\sin(x_1),$$

where $a, b \in \mathbb{R}$. We take $a = 7, b = 0.1, x_2 = x_3 = \pi/2$. Assume that $x_1 \sim \mathcal{U}(-\pi, \pi)$ and let $y(x_1)$ denote the stochastic version of $f(\boldsymbol{x})$. The polynomial chaos expansion of $y(x_1)$ reads

$$y(x_1) = \sum_{n=0}^{N-1} \hat{y}_n \Phi_n(x_1), \tag{1}$$

where

$$\hat{y}_n = \sum_{k=0}^{K-1} y(\zeta_k)\Phi_n(\zeta_k)\omega_k,$$

where $\{\zeta_k, \omega_k\}_{k=0}^{K-1}$ are suitably chose quadrature nodes and weights.

You are given the procedure PSA (see next page).

procedure PSA(N, K)
    generate K nodes and weights $\{\zeta_k, \omega_k\}_{k=0}^{K-1}$
    generate N orthogonal polynomials $\{\Phi_n\}_{n=0}^{N-1}$
    create a vector $\hat{y}$ to store the coefficients
    for $n \leftarrow 0, N-1$ **do**
        $\hat{y}_n \leftarrow 0$
        for $k \leftarrow 0, K-1$ **do**
            $\hat{y}_n \leftarrow \hat{y}_n + \text{IshigamiFunction}(\zeta_k)\Phi_n(\zeta_k)\omega_k$
        end for
    end for
end procedure

- Is the procedure PSA correctly implementing the pseudo-spectral approach?

- Would you implement PSA in your favorite programming language as given? Why? If you would implement it differently, sketch the modified procedure below.

(d) Assume that initially, the propagation of uncertainty though $y(x_1)$ was performed via standard Monte Carlo sampling using 10^6 samples. The corresponding values for the mean and variance are $\mathbb{E}[y(x_1)]_{MCS} = 7.001$, Var$[y(x_1)]_{MCS} = 1.294$.

Now, someone who did not participate in the *Algorithms for Uncertainty Quantification* course tries to use the polynomial chaos expansion + the pseudo-spectral approach to propagate the uncertainty through $y(x_1)$. The following questions come up:

- Once you asses the coefficients from Eq. (1), how do you compute the mean and variance?

- Assuming that a Gauss-Legendre quadrature rule with $K = 16$ was used to evaluate $N = 6$ coefficients. The results are

$$\hat{y}_0 = 7.000E00, \quad \hat{y}_1 = 4.890E-01, \quad \hat{y}_2 = -3.385E-15$$
$$\hat{y}_3 = -1.502E-01, \quad \hat{y}_4 = -1.580E-15, \quad \hat{y}_5 = 9.079E-03$$

Compute the mean and variance using the polynomial chaos coefficients. What do you observe when you compare these values with the Monte Carlo results? Why?

3 Intrusive Approaches (≈ 2 + 2 + 6 = 10 points)

(a) Name one advantage and one drawback of the stochastic Galerkin method.

(b) Consider a simplified version of the damped oscillator model problem

$$\begin{cases} \frac{d^2y}{dt^2}(t) + ky(t) = 1.0 \\ y(0) = y_0 \\ \frac{dy}{dt}(0) = y_1, \end{cases} \tag{2}$$

where k is uncertain and distributed as $k \sim \mathcal{N}(0, 4)$. Derive the polynomial chaos expansion of k.

(c) The stochastic solution $y(t, u_s)$ of the given problem (2) has to be approximated via a polynomial chaos approximation

$$y(t, u_s) = \sum_{n=0}^{N-1} \hat{y}_n(t)\Phi_n(u_s),$$

where the basis polynomials are orthonormal, i.e. $\mathbb{E}[\Phi_n \Phi_m] = \delta_{nm}$. Use the stochastic Galerkin approach to derive the system of equations needed to compute the coefficients $\hat{y}_n(t)$. *Hint: If you did not solve (b), take the polynomial chaos approximation of k to be $k = \hat{k}_0\Phi_0(u_s) + \hat{k}_1\Phi_1(u_s)$.*

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4 Global Sensitivity Analysis & Random Fields($\approx$ 1 + 1 + 2 + 2 = 6 points)

(a) What is the main goal of global Sensitivity Analysis?

(b) What technique allows to do a global Sensitivity Analysis in the context of UQ?

(c) What is a random field?

(d) If you encounter a problem involving a random field and you have to formulate the problem as a classical forward UQ problem, what option do you have?

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5 Software for UQ (≈ 4 points)

You are given a forward model with the following properties:

- a single run takes about 50 seconds.
- You know what phenomena the given code models, but you can use it only as a legacy code.
- 18 of its input parameters are uncertain.

What UQ methodology and which library/framework/toolkit would you use for the propagation of uncertainty? Why?