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Algorithms of Scientific Computing II

Exercise 7 - Smolyak-Quadrature and Jacobi

1 Smolyak Quadrature

In this assignment we will perform the Smolyak quadrature on the two dimensional domain $[0,1]^2$. We will use the trapezoidal rule as the nested univariate quadrature rule $Q_n^{(1)}$.

- Write a method, which returns for a given p and l the 1D-trapezoidal rule (weights) for p^l intervals.
- Write a function which implements the difference formula $\Delta_l^{(1)}$. The function takes as parameters p, l and a function to build the 1D quadrature rule (the above function) and returns a quadrature rule (weights).
- Next, implement the tensor product $\otimes$, which takes two 1D quadrature rules and builds a 2D rule.
- Now we can build the Smolyak quadrature by programming the sum on slide 97.

Jacobi-Iteration of $-u'' = f$

When solving a $n \times n$-SLE $Ax = b$ ($A \in R^{n\times n}$, $b, x \in R^n$) with a splitting method

$$Mx^{(i+1)} + (A - M)x^{(i)} = b,$$

we have following property for the error $e^{(i)} := x^{(i)} - x$. The linear projection of the error in each step $e^{(i)} \mapsto e^{(i+1)}$ is given by

$$e^{(i+1)} = Ne^{(i)}.$$

with

$$N := -M^{-1}(A - M) = I - M^{-1}A.$$

Eigenvalues and Eigenvectors of N are crucial for the convergence behavior. In general we don't know these values, but we will calculate them for a representative model-problem.

We start with the discretization of $-u'' = f$ on a grid with mesh width h. Therefore the system matrix is a tridiagonal matrix, with entries $2/h^2$ on the diagonal and $-1/h^2$ on the secondary diagonal. Please note, due to the tridiagonal shape a direct solver is very cheap, however we don't care about the solution right out since want to study the properties of splitting methods. In case of a multi-dimensional Laplace operator a direct solver is too expensive.

We are using a damped Jacobi with $0 < \alpha \leq 1$:

$$M := \frac{1}{\alpha} D_A.$$

Hereby, D_A gives A's diagonal.

A's eigenvectors and eigenvalues are given by:

$$\eta^k := \left(\sin\left(\frac{ik\pi}{m}\right)\right)_{1 \leq i < m} \in R^{m-1} \text{ and } \lambda_k := 4m^2 \sin^2\left(\frac{k\pi}{2m}\right)$$

a) By using A's eigenvectors and eigenvalues, please derive N's eigenvectors and eigenvalues.

   Now we can analyze the convergence behavior of the Jacobi method:

b) Which error-components (eigenvectors of N) vanish fast/slow?

c) Normally we start with random start solution $x^{(0)}$ and have therefore also a random mix of error-components. What's happening during the first iteration steps?

d) What's the convergence-rate of the „awkwardest" error-component?

e) At which iteration-count i do we have a guaranteed error-reduction $\|e^{(i)}\| \leq \varepsilon \|e^{(0)}\|$ for a given ε?

   How do things change when switching to a grid with halved mesh-width ($h/2$) (please assume $\sin(x) \approx x$ for small values of x)?