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Algorithms of Scientific Computing (Algorithmen des Wissenschaftlichen Rechnens)

Fast Poisson Solver

We will focus in this excercise on systems of linear equations, which are created, amongst others, during the discretization of differential equations (in this worksheet we will use the Poisson equation $-\Delta u = f$).

In the one-dimensional case we have the unknowns $u_1, \ldots, u_{N-1}$ in the equations

$$-u_{n+1} + 2u_n - u_{n-1} = f_n, \quad n = 1, \ldots, N-1 \tag{1}$$

For the first equation ($n = 1$) we take a virtual value $u_0 = 0$. Analog for the last equation ($n = N-1$) we take a $u_N = 0$ (In differential equations this complies with *homogenous Dirichlet boundary conditions*).

In the two-dimensional case we have the unknowns $u_{n,m}$ with $n, m = 1, \ldots N-1$ in the equations

$$-u_{n,m+1} - u_{n+1,m} + 4u_{n,m} - u_{n-1,m} - u_{n,m-1} = f_{n,m}, \quad n, m = 1, \ldots, N-1 \tag{2}$$

Here we need a whole bunch of virtual boundary values, which are all assumed to be zero:

$$u_{n,0} = u_{n,N} = u_{0,n} = u_{N,n} = 0, \quad n = 1, \ldots, N-1.$$

Excercise 1: Fast Poisson Solver for the 1d Problem

With the help of the Sine Transform an efficient solver for the system of equations (1) can be derived.

Therefore, we insert the transformed values

$$u_n = 2\sum_{k=1}^{N-1} U_k \sin\frac{\pi nk}{N}, \qquad f_n = 2\sum_{k=1}^{N-1} F_k \sin\frac{\pi nk}{N} \tag{3}$$

into the system of equations (1).

Show that you can set the coefficients U_k in the following way, depending on the F_k:

$$U_k = \frac{F_k}{2 - 2\cos\frac{\pi k}{N}} \qquad \text{for} \quad k = 1, \dots, N-1. \tag{4}$$

So, the U_k can be retrieved directly from the F_k, without solving a system of equations. Formulate an algorithm which solves the system of equations (1) efficiently by using the given dependency and the Fast Sine Transform(s).

Excercise 2: Fast Poisson Solver for the 2d Problem

In 1d the Fast Poisson Solver is not very helpfull, since the tridiagonal system of equations (1) can be solved by a Gauß elimination in $\mathcal{O}(N)$ operations. So we will try to bring this method to the 2d case.

Like in 1d we insert the transformed values

$$u_{n,m} = 2\sum_{k=1}^{N-1}\sum_{l=1}^{N-1} U_{k,l} \sin\frac{\pi nk}{N} \sin\frac{\pi ml}{N}, \qquad f_{n,m} = 2\sum_{k=1}^{N-1}\sum_{l=1}^{N-1} F_{k,l} \sin\frac{\pi nk}{N} \sin\frac{\pi ml}{N} \tag{5}$$

into the system of equations (2).

Show that this time you can set the $U_{k,l}$, depending on the $F_{k,l}$, like this:

$$U_{k,l} = \frac{F_{k,l}}{4 - 2\cos\frac{\pi k}{N} - 2\cos\frac{\pi l}{N}} \qquad \text{für} \quad k, l = 1, \dots, N-1 \tag{6}$$

So, again the $U_{k,l}$ can be retrieved directly form the $F_{k,l}$. Formulate an algorithm for 2d, which solves the system of equations (2) efficiently by using the Fast Sine Transform and the given dependency.