Algorithms of Scientific Computing (Algorithmen des Wissenschaftlichen Rechnens)

1 Hierarchical Basis of Polynomial Functions

After getting to know about the hierarchical basis consisting of linear hat functions last week we now introduce another hierarchical basis and compare the two of them with respect to cost and quality of approximation.

Our basis will consist of polynomial functions which will again be living in the unit interval. Level-wise construction of the support functions will work as follows:

- On level 0 there are the linear functions $p_{0,0}(x) = 1 - x$ and $p_{0,1}(x) = x$ with their degrees of freedoms (*dof*s) at $x = 0$ resp. $x = 1$.
- $p_{k,i}$ on level k will have children $p_{k+1,2i\pm1}$ on level $k + 1$ with the properties
  - $p_{k+1,2i\pm1}$ have the same roots as $p_{k,i}$ plus an additional root at $x = \frac{i}{2^k}$ (position of parent's *dof*)
  - $p_{k+1,2i\pm1}$ are 1 at positions of their *dof*s which are at $x = \frac{2i\pm1}{2^{k+1}}$
- $p_{k,i}$ is only defined and will only be evaluated in $[\frac{i-1}{2^k}, \frac{i+1}{2^k}]$
- $p_{0,0}$ and $p_{0,1}$ both have only one child, $p_{1,1}$, which is common to them

Questions:

1. What are the conditions that define support $p_{3,5}$?
2. Where is the *dof* located?
3. Name all its parents.
4. Compare the hierarchization routine to the one from last week. Where is additional effort required?
5. What is the global order of approximation?

2 Norms of Functions

When representing functions we will be interested in the question how "large" a function actually is. Measuring the difference between a function and its interpolant can for example help to make a statement about the quality of approximation.

We only consider functions $u : [0,1] \to \mathbb{R}$ with $u(0) = u(1) = 0$ and will mainly be interested in three norms:

- The infinity norm (German: Maximumsnorm)

$$\|u\|_\infty := \max_{x\in[0,1]} |u(x)|$$

- The L^2 norm

$$\|u\|_2 := \sqrt{\int_0^1 u(x)^2 \, dx},$$

  defined through the L^2 scalar product

$$(u,v)_2 := \int_0^1 u(x)v(x) \, dx$$

- The energy norm $\|u\|_E := \|u'\|_2$

Note: We always assume the existence of maxima, derivatives and integrals.

1. Compute these norms for

$$f_k(x) := \sin(k\pi x), \quad k \in \mathbb{N}$$

   and for

$$\phi_{l,i}(x) := \phi\left(2^l x - i\right) \quad l \in \mathbb{N}, i = 1, \ldots, 2^l - 1$$

   with $\phi(x) := \max\{1 - |x|, 0\}$.

2. For each of these norms prove the triangle inequality

$$\|u + v\| \leq \|u\| + \|v\|.$$

   For the L^2 norm use the Cauchy-Schwarz inequality

$$|(u,v)| \leq \|u\| \cdot \|v\|,$$

   that holds for arbitrary scalar products, i.e. also for the L^2 scalar product.