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# Algorithms of Scientific Computing (Algorithmen des Wissenschaftlichen Rechnens)

## Arithmetization of Space-Filling Curves – Solution

### Excercise 1: Calculation of $h$ for (in)finite fractions

Initially we calculate the decimal places of the numbers and get:

$$\begin{aligned} \frac{1}{8} &= 0_4.02 \\ \frac{1}{3} &= 0_4.11111111\ldots \end{aligned}$$

So, for $h\left(\frac{1}{8}\right)$ we get:

$$\begin{aligned} h\left(\frac{1}{8}\right) &= H_0 \circ H_2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} = H_0 \left( \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right) \\ &= H_0 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} \end{aligned}$$

The calculation of $h\left(\frac{1}{3}\right)$ turns out to be much more complicated, since we need to find the following limit:

$$h\left(\frac{1}{3}\right) = h(0_4.1111\ldots) = H_1 \circ H_1 \circ \ldots \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \lim_{n\to\infty} H_1^n \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So, we will write the operator $H_1$ in the following form:

$$H_1 = \underbrace{\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}}_{=:A_1} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{=:v} + \underbrace{\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}}_{=:b_1} = A_1 v + b_1.$$

From this, we get

$$\begin{array}{rcl} H_1^2 v & = & A_1(A_1 v + b_1) + b_1 = A_1^2 v + A_1 b_1 + b_1 \\ H_1^3 v & = & A_1(A_1^2 v + A_1 b_1 + b_1) + b_1 = A_1^3 v + A_1^2 b + A_1 b_1 + b_1 \\ \vdots & = & \vdots \\ H_1^n v & = & A_1^n v + A_1^{n-1} b_1 + \ldots + A_1 b_1 + b_1 \end{array}$$

For the term $A_1^{n-1} b_1 + \ldots + A_1 b_1 + b_1$ we use a trick, similiar to that used for geometric progressions:

$$\begin{array}{rcl} (I - A_1)\left(A_1^{n-1} b_1 + \ldots + A_1 b_1 + b_1\right) & = & A_1^{n-1} b_1 + \ldots + A_1 b_1 + b_1 \\ & & -A_1^n b_1 - A_1^{n-1} b_1 - \ldots - A_1 b_1 \\ & = & b_1 - A_1^n b_1 = (I - A_1^n)\, b_1 \end{array}$$

This leads to

$$A_1^{n-1} b_1 + \ldots + A_1 b_1 + b_1 = (I - A_1)^{-1}\,(I - A_1^n)\, b_1.$$

which renders to

$$\begin{array}{rcl} \lim\limits_{n\to\infty} H_1^n & = & \lim\limits_{n\to\infty} \left(A_1^n v + A_1^{n-1} b_1 + \ldots + A_1 b_1 + b_1\right) \\ & = & \lim\limits_{n\to\infty} \left( (I - A_1)^{-1}\,(I - \underbrace{A_1^n}_{\to 0}) b_1 \right) = (I - A_1)^{-1}\, b_1 \end{array}$$

I.e. we get the value $h\left(\frac{1}{3}\right) =: \binom{\xi}{\zeta}$ by solving the system of equations

$$(I - A_1)\begin{pmatrix}\xi \\ \zeta\end{pmatrix} = b_1 \qquad \Leftrightarrow \qquad \begin{pmatrix}\frac{1}{2} & 0 \\ 0 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\xi \\ \zeta\end{pmatrix} = \begin{pmatrix}0 \\ \frac{1}{2}\end{pmatrix}$$

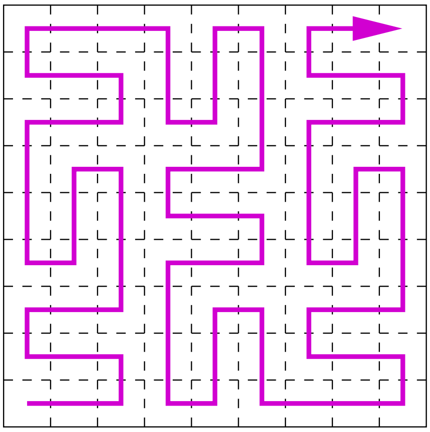
The solution $\xi = 0$ und $\zeta = 1$ can be found quite easily, so finally we get $h\left(\frac{1}{3}\right) = \binom{0}{1}$.

### Excercise 2: Arithmetization of the Peano Curve

Like for the arithmetization of the Hilbert curve, we assume that the parameter $t$ is given on the basis 9. $t = 0_9.n_1 n_2 n_3 n_4 \ldots$. Now we are looking for the operators $P_0, \ldots, P_8$, so that

$$p(0_9.n_1 n_2 n_3 n_4 \ldots) = P_{n_1} \circ P_{n_2} \circ P_{n_3} \circ P_{n_4} \circ \cdots \begin{pmatrix}0 \\ 0\end{pmatrix}.$$

For the operators $P_i$ and the algorithm for calculating the Peano functions refer to the Maple-worksheet `peano_switch.mws`.


(a) Peano curve

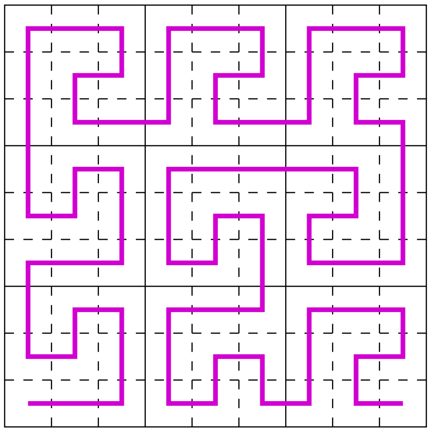

(b) meander-style Peano curve

Abbildung 1: Two space-filling curves of the Peano type

### Excercise 3: Koch Curve and approximating polgyongals

The implementation can be found in the Maple-worksheet `kochturtle.mw`.

### Excercise 4: Lengths and Areas

The length of an iteration of the Koch curve grows with each refinement step by the factor of $\frac{4}{3}$, since 3 subintervals are replaced by four sections of the same length. Accordingly is the length of the $n$th iteration $\left(\frac{4}{3}\right)^n \to \infty$. Thus, the Koch curve does not have a finite length.

The area that is included by the Koch curve grows by the area of triangles which are added in each iteration:

- In the first iteration the single equiliteral triangle has the side length $\frac{1}{3}$ and, thus, the area $\frac{1}{2} \cdot \frac{1}{3} \cdot \left(\frac{1}{3} \cdot \frac{1}{2}\sqrt{3}\right)$ ($\frac{1}{2}\sqrt{3}$ is the height of an equiliteral triangle with side length 1).
- With each iteration the area of one triangle shrinks by the factor of $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$.
- On the other hand the number of additional triangles grows in each iteration by the factor of 4.

So, in each iteration there is an additional area of $4^n \cdot \left(\frac{1}{9}\right)^n \cdot \frac{1}{36}\sqrt{3}$. Thus, in the limit we get a total area of

$$\sum_{n=1}^{\infty} \left(\frac{4}{9}\right)^n \cdot \frac{1}{36}\sqrt{3} = \frac{1}{1-\frac{4}{9}} \cdot \frac{1}{36}\sqrt{3} = \frac{1}{20}\sqrt{3}.$$