Technische Universität München | SS 2010
Lehrstuhl für Informatik V | Solution Worksheet 2
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Algorithms of Scientific Computing (Algorithmen des Wissenschaftlichen Rechnens)

1 Project: Interpolation of the Trajectory of the Asteroid Pallas[1] (Continued)

Excercise 1

$$
\begin{aligned}
X_l &= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} c_k e^{i2\pi kl/N} = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} c_k \cdot \left(\cos\left(\frac{2\pi kl}{N}\right) + i\sin\left(\frac{2\pi kl}{N}\right)\right) \\
&= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \Re\{c_k\} \cdot \left(\cos\left(\frac{2\pi kl}{N}\right) + i\sin\left(\frac{2\pi kl}{N}\right)\right) + i\Im\{c_k\} \cdot \left(\cos\left(\frac{2\pi kl}{N}\right) + i\sin\left(\frac{2\pi kl}{N}\right)\right)
\end{aligned}
$$

Now we sort for real and imaginery part of X_l:

$$
\begin{aligned}
\Re\{X_l\} &= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \Re\{c_k\} \cdot \cos\left(\frac{2\pi kl}{N}\right) - \Im\{x_k\} \cdot \sin\left(\frac{2\pi kl}{N}\right) \\
\Im\{X_l\} &= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \Im\{c_k\} \cdot \cos\left(\frac{2\pi kl}{N}\right) + \Re\{x_k\} \cdot \sin\left(\frac{2\pi kl}{N}\right)
\end{aligned}
$$

Since X_l is real $\Im\{X_l\}$ must be zero. Thus, the remaining part is

[1] Project idea and data are taken from: W. L. Briggs, Van Emden Henson, *The DFT – An Owner's Manual for the Discrete Fourier Transform*, SIAM, 1995

$$X_l = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \Re\{c_k\} \cdot \cos\left(\frac{2\pi kl}{N}\right) - \Im\{x_k\} \cdot \sin\left(\frac{2\pi kl}{N}\right)$$

By reducing the sum to the range $k = 1, \ldots, \frac{N}{2} - 1$, we get

$$X_l = \Re\{c_0\} + \Re\{c_{N/2}\} \cdot \cos(\pi l) + \sum_{k=1}^{\frac{N}{2}-1} \Re\{c_k\} \cdot \cos\left(\frac{2\pi kl}{N}\right) + \Re\{c_{-k}\} \cdot \cos\left(\frac{2\pi(-k)l}{N}\right) \\ - \Im\{c_k\} \cdot \sin\left(\frac{2\pi kl}{N}\right) - \Im\{c_{-k}\} \cdot \sin\left(\frac{2\pi(-k)l}{N}\right)$$

Using the symmetry of the sine and cosine functions, this can be derived to

$$X_l = \Re\{c_0\} + \Re\{c_{N/2}\} \cdot \cos(\pi l) + \sum_{k=1}^{\frac{N}{2}-1} (\Re\{c_k\} + \Re\{c_{-k}\}) \cdot \cos\left(\frac{2\pi kl}{N}\right) \\ + (\Im\{c_{-k}\} - \Im\{c_k\}) \cdot \sin\left(\frac{2\pi kl}{N}\right)$$

Since $c_{-k} = c_k^*$ it is $\Re\{c_{-k}\} = \Re\{c_k\}$ and $\Im\{c_{-k}\} = -\Im\{c_k\}$. So, we get

$$X_l = \Re\{c_0\} + \Re\{c_{N/2}\} \cdot \cos(\pi l) + \sum_{k=1}^{\frac{N}{2}-1} 2\Re\{c_k\} \cdot \cos\left(\frac{2\pi kl}{N}\right) - 2\Im\{c_k\} \cdot \sin\left(\frac{2\pi kl}{N}\right)$$

For $N = 12$, $a_k = 2\Re\{c_k\}, b_k = -2\Im\{c_k\}$ for all $k = 1, \ldots, \frac{N}{2}$, $a_0 = c_0$ and $a_{\frac{N}{2}} = c_{\frac{N}{2}}$ we get equation (2):

$$X_l = a_0 + \sum_{k=1}^{5} \left(a_k \cos\left(\frac{\pi kl}{6}\right) + b_k \sin\left(\frac{\pi kl}{6}\right)\right) + a_6 \cos(\pi l)$$

Excercise 2

The implementation is shown in the Maple worksheet uebung2.mw.

The function $C(x)$ is a complex function and in fact it is $\Im\{C(x)\} \neq 0$. Hence, it cannot be used for displaying the asteroid's trajectory.

The question is, if we can retrieve the trajectory from the $C(x)$, for example from $|C(x)|$ or $\Re\{C(x)\}$. We can find that it is $X(x) = \Re\{X(y)\}$, with $x = \frac{2Pi\theta}{180}$, i.e. the angel in radian.

$$\begin{aligned}
\Re\{C(x)\} &= \Re\left\{\sum_{k=-5}^{6} c_k e^{ikx}\right\} = \sum_{k=-5}^{6} \Re\left\{c_k e^{ikx}\right\} \\
&= \sum_{k=-5}^{6} \Re\Bigg\{ \Big(\Re\{c_k\} + i\Im\{c_k\}\big)(\cos(kx) + i\sin(kx)\Big) \Bigg\} \\
&= \sum_{k=-5}^{6} \Bigg(\Re\{x_k\}\cos(kx) - \Im\{c_k\}\sin(kx) \Bigg) \\
&= \Re\{c_0\} + \Re\{c_6\}\cos(6x) \\
&+ \sum_{k=-5}^{-1} \Bigg(\Re\{x_k\}\cos(kx) - \Im\{c_k\}\sin(kx) \Bigg) + \sum_{k=1}^{5} \Bigg(\Re\{x_k\}\cos(kx) - \Im\{c_k\}\sin(kx) \Bigg) \\
&= \Re\{c_0\} + \Re\{c_6\}\cos(6x) \\
&+ \sum_{k=1}^{5} \Bigg(\Re\{x_k\}\cos(kx) + \Re\{x_{-k}\}\cos(-kx) - \Im\{c_k\}\sin(kx) - \Im\{c_{-k}\}\sin(-kx) \Bigg)
\end{aligned}$$

Here we can use $c_{-k} = (c_k)^*$ and so $\Re\{c_{-k}\} = \Re\{c_k\}$ and $\Im\{c_{-k}\} = -\Im\{c_k\}$, respectivelly. In addition we set $a_0 = \Re\{c_0\}$, $a_6 = \Re\{c_6\}$, $a_k = 2\Re\{c_k\}$ and $b_k = -2\Im\{c_k\}$, for $k = 1, \ldots, 5$. So, we get

$$\begin{aligned}
\Re\{C(x)\} &= \Re\{c_0\} + \Re\{c_6\}\cos(6x) \\
&+ \sum_{k=1}^{5} \Bigg(\Re\{x_k\}\cos(kx) + \Re\{x_{-k}\}\cos(-kx) - \Im\{c_k\}\sin(kx) - \Im\{c_{-k}\}\sin(-kx) \Bigg) \\
&= \Re\{c_0\} + \Re\{c_6\}\cos(6x) \\
&+ \sum_{k=1}^{5} \Bigg(\Re\{c_k\}\cos(kx) + \Re\{c_k\}\cos(kx) - \Im\{c_k\}\sin(kx) - \Im\{c_k\}\sin(kx) \Bigg) \\
&= a_0 + a_6\cos(6x) + \sum_{k=1}^{5} \Big(2\Re\{c_k\}\cos(kx) - 2\Im\{c_k\}\sin(kx) \Big) \\
&= a_0 + \sum_{k=1}^{5} \Big(a_k\cos(kx) + b_k\sin(kx) \Big) + a_6\cos(6x) \\
&= X(x)
\end{aligned}$$

Thus, the real part of function $C(x)$ is the same as the real interpolation function $X(x)$.

However: It is **not** $C(x) = X(x)$, since $\Im\{C(x)\} \neq 0$.

Additional Excercises

Excercise 3

Let ω_N be $\omega_N := e^{i2\pi/N}$.

ω_N^k is the solution of the equation $z^N = 1$ (for all $k \in \mathbb{Z}$)

Assume $z := \omega_N^k$, thus:

$$z^N = (\omega_N^k)^N = \left(e^{\frac{i2\pi k}{N}}\right)^N = e^{\frac{i2\pi Nk}{N}} = e^{i2\pi k} = \underbrace{\cos(2\pi k)}_{=1 \text{ for all } k\in\mathbb{Z}} + i \underbrace{\sin(2\pi k)}_{=0 \text{ for all } k\in\mathbb{Z}} = 1$$

The complex conjugates hold $(\omega_N^k)^* = \omega_N^{-k}$

$$(\omega_N^k)^* = \left(e^{\frac{i2\pi k}{N}}\right)^* = \left(\cos\left(\frac{2\pi k}{N}\right) + i\sin\left(\frac{2\pi k}{N}\right)\right)^* = \left(\cos\left(\frac{2\pi k}{N}\right) - i\sin\left(\frac{2\pi k}{N}\right)\right)$$
$$= \left(\cos\left(\frac{2\pi(-k)}{N}\right) + i\sin\left(\frac{2\pi(-k)}{N}\right)\right) = e^{\frac{i2\pi(-k)}{N}} = \omega_N^{-k}$$

Periodicity

$$\left\{\omega_N^k\right\}_{k=-\infty}^{\infty} = \left\{e^{\frac{i2\pi k}{N}}\right\}_{k=-\infty}^{\infty} = \left\{\cos\left(\frac{2\pi k}{N}\right) + i\sin\left(\frac{2\pi k}{N}\right)\right\}_{k=-\infty}^{\infty}$$

Since both, $\cos\left(\frac{2\pi k}{N}\right)$ and $\sin\left(\frac{2\pi k}{N}\right)$ are N-periodic in $k = -\infty \ldots \infty$, also the sum of them must be N-periodic.

Assume a fixed n. Then

$$\left\{\omega_N^{nk}\right\}_{k=-\infty}^{\infty} = \left\{\cos\left(\frac{2\pi nk}{N}\right) + i\sin\left(\frac{2\pi nk}{N}\right)\right\}_{k=-\infty}^{\infty}$$
$$= \left\{\cos\left(n\cdot\frac{2\pi k}{N}\right) + i\sin\left(n\cdot\frac{2\pi k}{N}\right)\right\}_{k=-\infty}^{\infty}$$

If $\cos\left(\frac{2\pi k}{N}\right)$ is N-periodical, $\cos\left(n\cdot\frac{2\pi k}{N}\right)$ is $\frac{N}{n}$-periodical and, thus also N-periodical, for $n = -N, \ldots -1, 1, \ldots, N$.

If $n = 0$ then $\cos\left(n \cdot \frac{2\pi k}{N}\right) = \cos(0) = 1$, which is trivially N-periodical.

If $|n| > N$ then there is a $c \in \mathbb{Z}$, so that $n := n - c \cdot N$ with $|n'| < N$. Then

$$\begin{aligned}\cos\left(n \cdot \frac{2\pi k}{N}\right) &= \cos\left((cN + n') \cdot \frac{2\pi k}{N}\right) = W \cos\left(\frac{2\pi k N}{N} + \frac{2\pi k n'}{N}\right) = \cos\left(2\pi k + \frac{2\pi k n'}{N}\right) \\ &= \cos\left(\frac{2\pi k n'}{N}\right)\end{aligned}$$

which is again N-periodical, as shown above. The same can be shown for a fixed k.

Excercise 4

The proof follows directly from the last excercise:

$$f_{n+N} = \sum_{k=0}^{N-1} F_k \omega_N^{(n+N)k} = \sum_{k=0}^{N-1} F_k \omega_N^{nk} = f_n$$

and

$$F_{k+N} = \frac{1}{N} \sum_{n=0}^{N-1} f_n \omega_N^{-n(k+N)} = \frac{1}{N} \sum_{n=0}^{N-1} f_n \omega_N^{-nk} = F_k$$