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SS 2010
Solution Worksheet 3
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Algorithms of Scientific Computing (Algorithmen des Wissenschaftlichen Rechnens)

– Solution –

Excercise 1: DFT and „Padding"

For the classic Fast Fourier Transform the number of discrete data must be a power of two. If this is not the case, one could try to fill up the dataset by "zero" entries like this:

$$\hat{f}_n := \begin{cases} f_n & \text{if} \quad n \leq N-1 \\ 0 & \text{if} \quad N \leq n \leq M-1 \end{cases}$$

The Fourier coefficients $\widehat{F}_k$ of the extended dataset then add up to

$$\widehat{F}_k = \frac{1}{M} \sum_{n=0}^{M-1} \hat{f}_n \omega_M^{-kn} = \frac{1}{M} \sum_{n=0}^{N-1} f_n \omega_M^{-kn}.$$

This looks like if the $\widehat{F}_k$ are just the $\frac{N}{M}$ multiple of the original coefficients from the transform of length N:

$$F_k = \frac{1}{N} \sum_{n=0}^{N-1} f_n \omega_N^{-kn}.$$

However, this is not the case, since

$$\omega_N^{-kn} \neq \omega_M^{-kn}.$$

So, the frequencies of the base functions do change.

If we take the Fourier transform as an interpolation problem, then the extention of the dataset is equal to an increment of the number of supporting points. Since the observed interval stays the same ($[0, 2\pi]$), the distance between the supporting points must decrease. By padding the dataset with "zeros" we actually compressed the signal and therefore the signal must be assembled from higher-frequency oscillations.

Wir setzen die obige Rechnung fort, indem wir zunächst berechen, dass

We go on with the equation from above. First we show that

$$\omega_M^{-kn} = e^{-i2\pi kn/M} = e^{-i2\pi kn(N/M)/N} = \left(\omega_N^{-kn}\right)^{N/M}$$

holds and therefore

$$\widehat{F}_k = \frac{1}{M}\sum_{n=0}^{N-1} f_n \left(\omega_N^{-kn}\right)^{N/M} = \frac{N}{M}\cdot\frac{1}{N}\sum_{n=0}^{N-1} f_n \omega_N^{-k(N/M)n}$$

In general we cannot express this by the F_k. But if kN/M is an integer number, we get

$$\widehat{F}_k = \frac{N}{M}\cdot\frac{1}{N}\sum_{n=0}^{N-1} f_n \omega_N^{-k(N/M)n} = \frac{N}{M} F_{kN/M}$$

Explanation: The Fourier components $\widehat{F}_k$ of the compressed signal belong to the wavenumber k. In the original signal the same component would belong to the oscillation with wavenumber kN/M. If kN/M is an integer number, this Fourier component is also computed in the "short" transformation and can be taken from the "long" transformation directly without being changed. If kN/M is not an integer number, then there is no according component in the "short" transformation.

Excercise 2: Discrete Cosine Transform

The proof is done in the following steps:

① Isolate the symmetry condition

② Insert the symmetry condition

③ Assemble terms to a sum over f_n

④ Make terms "real"

$$\begin{aligned} F_k &= \frac{1}{2N}\sum_{n=-N+1}^{N} f_n \omega_{2N}^{-kn} \\ &\overset{①}{=} \frac{1}{2N}\left(\sum_{n=-N+1}^{-1} f_n \omega_{2N}^{-kn} + f_0\omega_{2N}^{0} + \sum_{n=1}^{N-1} f_n \omega_{2N}^{-kn} + f_N \omega_{2N}^{-kN}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2N}\left(\sum_{n=1}^{N-1} f_{-n}\omega_{2N}^{kn} + f_0 e^0 + \sum_{n=1}^{N-1} f_n \omega_{2N}^{-kn} + f_N e^{-i2\pi kN/2N}\right)\\
&\overset{②}{=} \frac{1}{2N}\left(\sum_{n=1}^{N-1} f_n\omega_{2N}^{kn} + f_0 + \sum_{n=1}^{N-1} f_n \omega_{2N}^{-kn} + f_N e^{-i\pi k}\right)\\
&\overset{③}{=} \frac{1}{2N}\left(f_0 + \sum_{n=1}^{N-1} f_n \underbrace{\left(\omega_{2N}^{kn} + \omega_{2N}^{-kn}\right)}_{=\omega_{2N}^{kn}+\left(\omega_{2N}^{kn}\right)^*=2\mathrm{Re}\{\omega_{2N}^{kn}\}} + f_N e^{-i\pi k}\right)\\
&\overset{④}{=} \frac{1}{2N}\left(f_0 + 2\sum_{n=1}^{N-1} f_n \underbrace{\mathrm{Re}\{e^{i2\pi kn/2N}\}}_{=\mathrm{Re}\{\cos\left(\frac{\pi kn}{N}\right)+i\sin\left(\frac{\pi kn}{N}\right)\}} + f_N\left(\cos(-\pi k) + i\sin(-\pi k)\right)\right)\\
&= \frac{1}{N}\left(\frac{1}{2}f_0 + \sum_{n=1}^{N-1} f_n \cos\left(\frac{\pi kn}{N}\right) + \frac{1}{2} f_N \cos(\pi k)\right) \in \mathbb{R} \qquad \text{q.e.d.}
\end{aligned}
$$

b) Show that the F_k also have a symmetry:

Since there are only cosine terms we assume (and get) an even symmetry:

$$
\begin{aligned}
F_{-k} &= \frac{1}{N}\left(\frac{1}{2}f_0 + \sum_{n=1}^{N-1} f_n \cos\left(\frac{-\pi kn}{N}\right) + \frac{1}{2} f_N \cos(\pi n)\right)\\
&= \frac{1}{N}\left(\frac{1}{2}f_0 + \sum_{n=1}^{N-1} f_n \cos\left(\frac{\pi kn}{N}\right) + \frac{1}{2} f_N \cos(\pi n)\right)\\
&= F_k
\end{aligned}
$$

Since all $F_{-k} = F_k$, we need the F_k only for $k = 0, \ldots, N$ for a Cosine Transform.

c) Algorithm for the Cosine Transform

The procedure `FFT(f,N)` computes the correct coefficients, if we pass the $N + 1$ data from field g as a dataset of length $2N$ with symmetry $f_{-n} = f_n$.

From equation (1) of the worksheet we know that `FFT(f,N)` gets a dataset `f` with indices $n = -N + 1, \ldots, N$. We only have to compute the F_k for $k = 0, \ldots, N$.

So, the algorithm looks like this:

1. For all $n = 0, \ldots, N$:
   Set `f[n] := g[n]` $= f_n$
   Set `f[-n] := g[n]` $= f_n$
2. Call `FFT(f,`N`)`
3. *(Now the Fourier coefficients F_k are stored in the field f)*
   For all $k = 0, \ldots, N$:
   Set `g[k] := f[k]` $= F_k$

Excercise 3: Fast Discrete Cosine Transform

The butterfly scheme is retrieved as usual:

$$
\begin{aligned}
F_k &= \frac{1}{2N}\sum_{n=-N+1}^{N} f_n\omega_{2N}^{-kn} = \frac{1}{2}\left(\frac{1}{N}\sum_{n=-N/2+1}^{N/2} f_{2n}\omega_{2N}^{-2kn} + \frac{1}{N}\sum_{n=-N/2+1}^{N/2} f_{2n-1}\omega_{2N}^{-k(2n-1)}\right)\\
&= \frac{1}{2}\Bigg(\underbrace{\frac{1}{N}\sum_{n=-N/2+1}^{N/2} f_{2n}\omega_{N}^{-kn}}_{=:G_k} + \underbrace{\frac{1}{N}\sum_{n=-N/2+1}^{N/2} f_{2n-1}\omega_{N}^{-kn}}_{=:H_k}\omega_{2N}^{k}\Bigg)\\
&= \frac{1}{2}\left(G_k + \omega_{2N}^{k}H_k\right)\\
F_{k+N} &= \frac{1}{2}\left(G_{k+N} + \omega_{2N}^{k+N}H_{k+N}\right) = \frac{1}{2}\left(G_k - \omega_{2N}^{k}H_k\right)
\end{aligned}
$$

For the datasets $g_n := f_{2n}$ and $h_n := f_{2n-1}$, respectively, we can try to find other symmetries:

$$g_{-n} = f_{2(-n)} = f_{-2n} = f_{2n} = g_n$$

The "even" data also shows an even symmetry and therefore lead to another Cosine Transform but with half length.

Analog for the data with odd indices:

$$h_{-n} = f_{2(-n)-1} = f_{-2n-1} = f_{2n+1} = f_{2(n+1)-1} = h_{n+1}$$

Again we get an "even" symmetry. However, this is the transform shown in the lecture, known as Quarter-Wave-DCT, again with half length.