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Lehrstuhl für Informatik V
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SS 2010
Sheet 4
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Algorithms of Scientific Computing
(Algorithmen des Wissenschaftlichen Rechnens)

Hierarchical Surplus

Proposed solution

- Write two functions that implement the mapping from the u_i to the v_i and vice versa. The mapping is supposed to happen "in place", i.e. no additional storage is to be allocated (except from local variables).

  ref. to maple worksheet

- Test your functions for values $u_i := x_i^N$, $N = 1, 2, 3$.

  ref. to maple worksheet

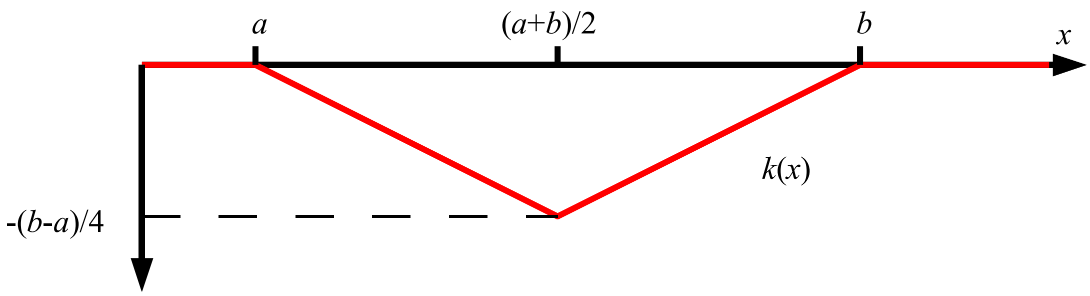
- Show that for the hierarchical surplus

$$\Delta = u\left(\frac{a+b}{2}\right) - \frac{u(a)+u(b)}{2}$$

  of a sufficiently often continuously differentiable function $u : [a,b] \to \mathbb{R}$ the following condition holds:

$$\Delta = \int_a^b k(x) \cdot u''(x)dx, \quad \text{with } k(x) = \begin{cases} \frac{1}{2}(a-x) & \text{for } a \leq x < \frac{a+b}{2} \\ \frac{1}{2}(x-b) & \text{for } \frac{a+b}{2} \leq x < b \\ 0 & \text{else.} \end{cases}$$

  The function plot of $k(x)$ looks like this



We get the identity after applying partial integration twice

$$\int_{x_0}^{x_1} f \cdot g' = f \cdot g|_{x_0}^{x_1} - \int_{x_0}^{x_1} f' \cdot g,$$

during which we associate k with the derivatives of u.

The positions with the cracks require extra treatment so we divide the global integral into two parts with

$$\underbrace{\int_a^m k(x) \cdot u''(x)\, dx}_{I} + \underbrace{\int_m^b k(x) \cdot u''(x)\, dx}_{II}$$

First partial integration ($f := k$, i.e. $f' = -1/2$ and $g := u'$):

$$I = \underbrace{k(x)u'(x)|_a^m}_{-\frac{h}{2}u'(m)} - \int_a^m \left(-\frac{1}{2}\right) u'(x)\, dx$$

Second partial integration ($f := -1/2$, i.e. $f' = 0$ and $g := u''$):

$$\begin{aligned} \ldots &= -\frac{h}{2}u'(m) - \left(-\frac{1}{2}\right) u(x)\bigg|_a^m \\ &= -\frac{h}{2}u'(m) + \frac{1}{2}u(m) - \frac{1}{2}u(a). \end{aligned}$$

And just the same for the part on the right hand side

$$\begin{aligned} II &= \underbrace{k(x)u'(x)|_m^b}_{\frac{h}{2}u'(m)} - \int_m^b \left(\frac{1}{2}\right) u'(x)\, dx \\ &= \frac{h}{2}u'(m) - \left(\frac{1}{2}\right) u(x)\bigg|_m^b \\ &= \frac{h}{2}u'(m) + \frac{1}{2}u(m) - \frac{1}{2}u(b). \end{aligned}$$

The sum of both gives us the result we were looking for

$$I + II = u(m) - \frac{u(a) + u(b)}{2}.$$