Technische Universität München
Lehrstuhl für Informatik V
Dr. T. Neckel, D. Pflüger

SS 2010
Sheet 5
2.6.2010

Algorithms of Scientific Computing
(Algorithmen des Wissenschaftlichen Rechnens)
Hierarchical Basis and Norms of Functions
Proposed solution

1 Hierarchical Basis of Polynomial Functions

(i)—(iii) Have a look at the maple worksheet (or the exported html version of it)

(iv) More effort is necessary when making recursive calls to compute the surpluses of higher levels. Since linear interpolation of "low level contribution" is not possible any more we need to pass more information by means of formal parameters, e.g. the explicit forms of the accumulated lower level polynomials as coefficient vectors.
The implementation in the worksheet is inefficient with regard to the number and the style of polynomial evaluations (e.g. *Horner scheme*).

(v) Like the hat function representation the global approximation is only C_0-continuous, too, while the order of approximation is higher (introducing pol. degree p on a new level scales the error in the order of $\left(\frac{1}{\sqrt{2}}\right)^{p+1}$).
As this is only one example for a hierarchical basis of polynomials, a different set of conditions for the construction leads to a different basis. Thus we can also increase the global degree of continuity by introducing spline-like conditions for a higher level of smoothness.
Remark: In practice polynomial degress $p \geq 7$ are usually not used.

2 Norms of Functions

1a) $f_k(x) := \sin(k\pi x), \quad k \in \mathbb{N}$

- $\|f_k\|_\infty = 1$ — the only thing interesting is that for every $k > 0$ the function actually assumes this maximum.

- Now we need the antiderivative of f_k^2 (look up, ask maple, partial integration,...):

$$\frac{x}{2} - \frac{sin(k\pi x)cos(k\pi x)}{2k\pi},$$

(don't believe it? take the derivative!). The result is

$$\int_0^1 f_k(x)^2\,dx = \frac{1}{2} \qquad \text{and thus} \qquad \|f_k\|_2 = \sqrt{\frac{1}{2}}$$

(independent of k).

- It works similar for the energy norm since the antiderivative of $(f_k')^2(x) = (k\pi\cos(k\pi x))^2$ is

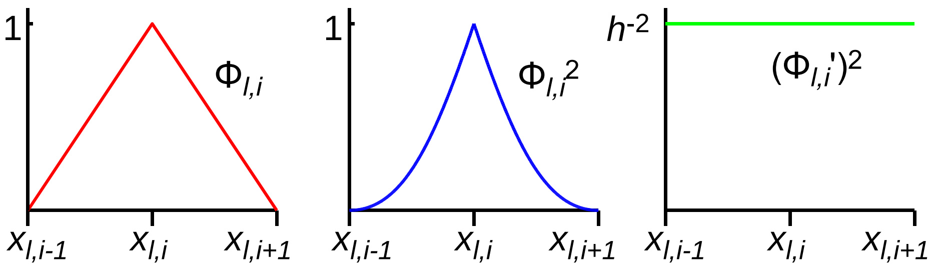
$$(k\pi)^2 \cdot \left(\frac{x}{2} + \frac{sin(k\pi x)cos(k\pi x)}{2k\pi}\right).$$

Apparently in the energy norm

$$\|f_k\|_E = k\pi\sqrt{\frac{1}{2}}.$$

higher frequencies have a stronger influence.

1b) $\phi_{l,i}(x) := \phi\left(2^l x - i\right)$ Integration is easier here, after one has understood what the functions look like (with $x_{l,i} := i \cdot 2^{-l}$ and $h = 2^{-l}$):



- $\|\phi_{l,i}\|_\infty = 1$ (by mere looking) independent of l and i.
- At first we compute the L^2 norm for $\phi = \phi_{0,0}$, i.e. for once we consider $[-1,1]$ instead of $[0,1]$. We get

$$\int_{-1}^1 \phi(x)^2\,dx = 2\int_0^1 x^2\,dx = 2\left[\frac{x^3}{3}\right]_{x=0}^1 = \frac{2}{3},$$

and thus $\|\phi\|_2 = \sqrt{2/3}$ (still in $[-1,1]$!).

In order to transform ϕ to $\phi_{l,i}$ we translate it (no change to the integral) and scale it by a factor $h = 2^{-l}$. Taking the scaling into consideration the norm can be rewritten as

$$\|\phi_{l,i}\|_2 = \sqrt{\frac{2}{3}2^{-l}}.$$

"Small looking" basis functions apparently are also less important.

- The picture on the right allows us to directly determine the energy norm (the quadrangle has width $2 \cdot 2^{-l}$ and height 2^{2l}):

$$\|\phi_{l,i}\|_E = \sqrt{2 \cdot 2^l}.$$

Here, "small looking" basis functions are *more* important!

2) For each of these norms prove the *triangle inequality*

$$\|u + v\| \leq \|u\| + \|v\|.$$

For the L^2 norm use the Cauchy-Schwarz inequality

$$|(u, v)| \leq \|u\| \cdot \|v\|,$$

that holds for arbitrary scalar products, i.e. also for the L^2 scalar product.

- Infinity norm: Let $x \in [0, 1]$ such that $\|u + v\|_\infty = |u(x) + v(x)|$. We directly get

$$\|u + v\|_\infty = |u(x) + v(x)| \leq |u(x)| + |v(x)| \leq \|u\|_\infty + \|v\|_\infty$$

(for both "$\leq$" the case of "$<$" is possible, but that's not important).

- L^2 norm:

$$\begin{aligned}
\|u + v\|_2^2 &= (u + v, u + v)_2 \\
&= (u, u)_2 + (u, v)_2 + (v, u)_2 + (v, v)_2 \\
&\leq (u, u)_2 + 2|(u, v)_2| + (v, v)_2 \\
&\leq (u, u)_2 + 2\|u\|_2\|v\|_2 + (v, v)_2 \\
&= (\|u\| + \|v\|)^2.
\end{aligned}$$

- Energy norm: see definition and previous item