

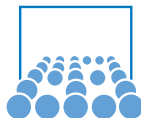
# Algorithms of Scientific Computing

## Hierarchical Methods and Sparse Grids

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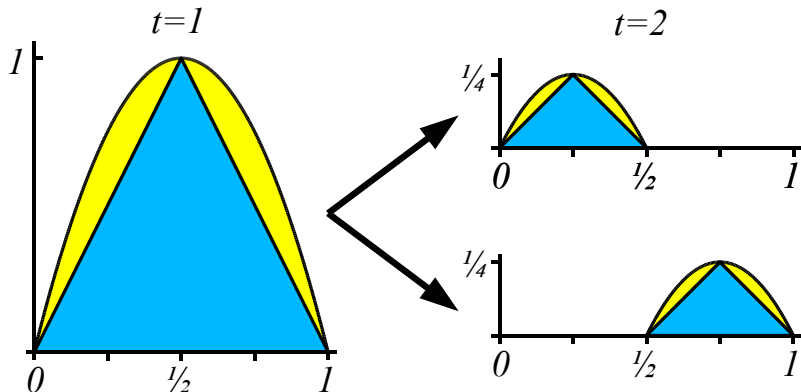


## Part III

# Hierarchical Decomposition, 1d

# Archimedes' Quadrature

Compute an approximation of  $F_1 := \int_0^1 4 \cdot x \cdot (1-x) dx = \frac{2}{3}$



## Archimedes' Quadrature (2)

- Integrating  $4x(1-x)$ , we have to consider several quantities
- Ordered by (recursive) level  $t$ :

Level-depth	1	2	3	4	...	$t$
Mesh-width $h$	$1/2$	$1/4$	$1/8$	$1/16$	...	$2^{-t}$
# triangles	1	2	4	8	...	$\frac{1}{2}2^t$
surplus $v$	1	$1/4$	$1/16$	$1/64$	...	$4 \cdot 2^{-2t}$
Area of triangle $D_1$	$1/2$	$1/16$	$1/128$	$1/1024$	...	$4 \cdot 2^{-3t}$
Sum (current $t$ )	$1/2$	$1/8$	$1/32$	$1/128$	...	$2 \cdot 2^{-2t}$
Sum ( $\leq t$ )	$1/2$	$5/8$	$21/32$	$85/128$	...	$\frac{2}{3}(1 - 2^{-2t})$
Error	$1/6$	$1/24$	$1/96$	$1/384$	...	$\frac{2}{3}2^{-2t}$

# Approximation of Functions

- To analyze Archimedes' quadrature rule, we consider functions
- We need a representation of the (approximating) function  $u(x)$  which we are integrating:
  - $u$  as linear combination of ansatz functions  $\phi_j$ :

$$u(x) = \sum_{i=1}^n \alpha_i \cdot \phi_i(x)$$

- Integrating  $u(x)$ :

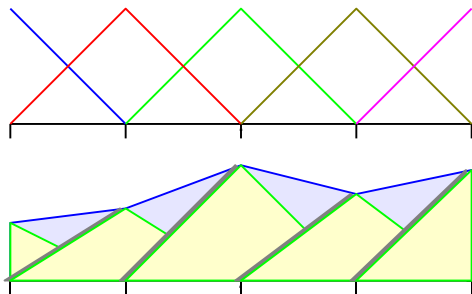
$$\int_a^b u(x) dx = \sum_i^n \alpha_i \int_a^b \phi_i(x) dx,$$

- Weighted sum of  $\alpha_j$
- Remember: Newton-Cotes formulas are weighted sum of function evaluations

# Composite Trapezoidal Rule: Function

## Interpolant

- Continuous, piecewise linear function
- Represent  $u$  in nodal point (hat) basis



- Koefficients  $\alpha_j$  are function values at grid points
- Ansatz functions have area  $h$  ( $h/2$  at boundaries)

# Piecewise Linear Functions

## Ansatz space

- Only consider  $u : [0, 1] \rightarrow \mathbb{R}$
- Consider discretization level  $n \in \mathbb{N}$
- Obtain
  - Mesh-width  $h_n = 2^{-n}$
  - Grid points  $x_{n,i} = i \cdot h_n$
  - Define “mother of all hat functions”

$$\phi(x) := \max\{1 - |x|, 0\}$$

⇒ Ansatz functions

$$\phi_{n,i}(x) := \phi\left(\frac{x - x_{n,i}}{h_n}\right)$$

- Nodal point basis  $\Phi_n := \{\phi_{n,i}, 0 \leq i \leq 2^n\}$

## Piecewise Linear Functions (2)

- Space of continuous piecewise linear functions

$$V_n = \text{span}(\Phi_n)$$

- Interpolants  $u_n \in V_n$

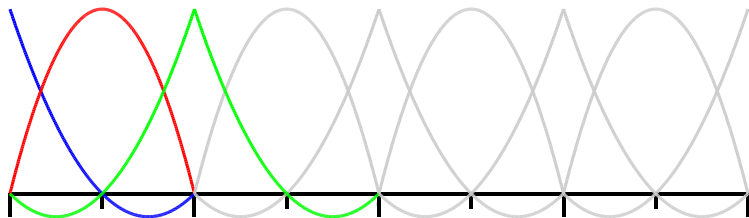
$$u_n(x) = \sum_{i=0}^{2^n} \alpha_{n,i} \phi_{n,i}(x)$$



# Composite Simpson's Rule: Function

## Interpolant

- Continuous, piecewise quadratic function
- More complicated basis:



- Ansatz functions: Lagrangian polynomials, glued together
- $\alpha_j$ : function values at grid points
- Ansatz functions have area  $h/6$  (blue),  $4h/6$  (red),  $2h/6$  (green)
- We'll not formally define basis functions here...

# From Composite Trapezoidal to Archimedes

## Piecewise linear functions

- We restrict our functions  $u$  to  $u(0) = u(1) = 0$
- Nodal point basis for discretization level  $n$ :

$$\Phi_n := \{\phi_{n,i}, 1 \leq i \leq 2^n - 1\}$$

- Function space

$$V := \bigcup_{l=1}^{\infty} V_l$$

contains all functions which are for sufficiently large  $l$  in  $V_l$

- Generating system of  $V$  as

$$\Phi := \bigcup_{l=1}^{\infty} \Phi_l$$

- Note: not minimal, thus not a basis (not linear independent)

# Hierarchical Basis

- We are interested in a hierarchical decomposition of  $V_l$ 
  - Define *hierarchical increment*  $W_l$ , s.t.  $V_l$  is a *direct sum* of  $W_l$ :

$$V_l = V_{l-1} \oplus W_l$$

## Side-note: direct sum

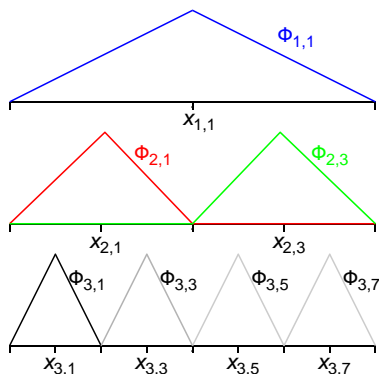
- Every  $u_l \in V_l$  can be uniquely decomposed as  $u_l = u_{l-1} + w_l$ , with  $u_{l-1} \in V_{l-1}$  and  $w_l \in W_l$
- $W_l$  has to contain  $2^{l-1}$  ansatz functions:  
 $\dim V_l = 2^l - 1 = \dim V_{l-1} + \dim W_l$
- This holds (introducing index sets  $I_l$ ) for

$$I_l := \{i : 1 \leq i < 2^l, i \text{ odd}\}$$

$$W_l := \text{span} \{\phi_{l,i} : i \in I_l\}$$

# Hierarchical Increments

- Set of hierarchical increments  $W_l$
- For  $l = 1$ :  $W_1 = V_1$
- Example for  $l = 1, 2, 3$ :



## Hierarchical Basis (cont.)

- Then

$$V_n = \bigoplus_{l=1}^n W_l$$

is a direct sum, too:

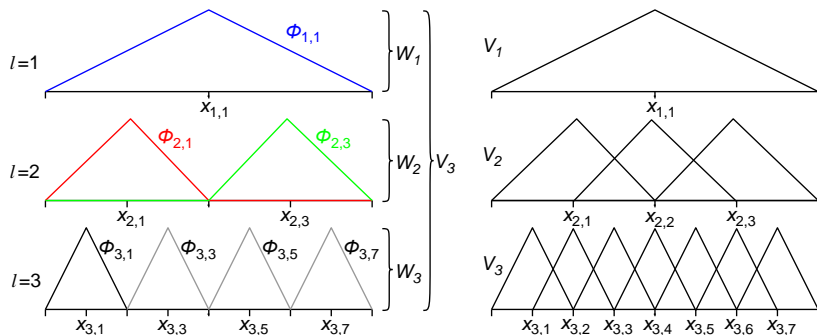
- $u \in V_n$  can be decomposed uniquely into  $w_l \in W_l$ :

$$u = \sum_{l=1}^n w_l = \sum_{l=1}^n \sum_{i \in I_l} v_{l,i} \phi_{l,i}$$

- Coefficients  $v_{l,i}$  are hierarchical surplusses
- Corresponding basis of  $V_n$  (or, with  $\infty$  instead of  $n$ , of  $V$ )

$$\Psi_n := \bigcup_{l=1}^n \{\phi_{l,i} : i \in I_l\}.$$

# Comparison



# Analysis of Hierarchical Decomposition

- Contribution of summands in hierarchical decomposition

$$u = \sum_{l=1}^n w_l = \sum_{l=1}^n \sum_{i \in I_l} v_{l,i} \phi_{l,i}.$$

- Interesting in univariate setting
- Will be crucial in multivariate setting
  - Cost/benefit analysis will help to significantly reduce effort
- Need several norms to measure  $w_l$  (cf. worksheet 5)

# Norms of Functions

Again, we assume sufficiently smooth functions  $u : [0, 1] \rightarrow \mathbb{R}$

## Norms

- Maximum-norm

$$\|u\|_{\infty} := \max_{x \in [0, 1]} |u(x)|$$

- $L^2$ -norm

$$\|u\|_2 := \sqrt{\int_0^1 u(x)^2 dx},$$

for the  $L^2$  scalar product

$$(u, v)_2 := \int_0^1 u(x)v(x) dx$$

- Energy-norm

$$\|u\|_E := \|u'\|_2$$



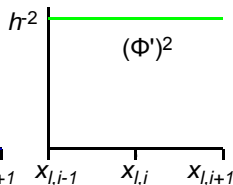
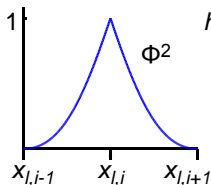
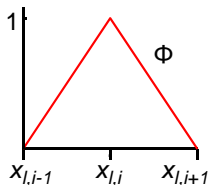
# Norms of Basis Functions

For the basis functions  $\phi_{l,i}$ , we obtain

$$\|\phi_{l,i}\|_{\infty} = 1$$

$$\|\phi_{l,i}\|_2 = \sqrt{\frac{2h_l}{3}}$$

$$\|\phi_{l,i}\|_E = \sqrt{\frac{2}{h_l}}$$



# Estimation of Surplusses

- Let  $\psi_{l,i} := -\frac{h_l}{2} \phi_{l,i}$
  - Surplus  $v_{l,i}$  of basis function  $\phi_{l,i}$
  - $u$  two times differentiable
- ⇒ We can then write  $v_{l,i}$  as (compare worksheet 4)

$$v_{l,i} = \int_0^1 \psi_{l,i}(x) u''(x) dx.$$

- $v_{l,i}$  depends on  $u''$ , thus we define for future use

$$\mu_2(u) := \|u''\|_2 \quad \text{und} \quad \mu_\infty(u) := \|u''\|_\infty.$$

## Estimation of Surplusses (2)

- Starting from integral representation of  $v_{l,i}$ , we can bound

$$|v_{l,i}| \leq \frac{h_l}{2} \cdot \left( \int_0^1 \phi_{l,i} dx \right) \cdot \mu_\infty(u) = \frac{h_l^2}{2} \cdot \mu_\infty(u)$$

- and (via Cauchy-Schwartz inequality  $|(u, v)| \leq \|u\| \cdot \|v\|$ )

$$|v_{l,i}| \leq \frac{h_l}{2} \|\phi_{l,i}\|_2 \cdot \mu_2(u|_{T_i}) = \sqrt{\frac{h_l^3}{6}} \cdot \mu_2(u|_{T_i}),$$

- $u|_{T_i}$  restricts  $u$  to the support  $T_i = [x_{l,i-1}, x_{l,i+1}]$  of  $\phi_{l,i}$

# Estimation of $w_I$

- Estimate contribution of

$$w_I = \sum_{i \in I_I} v_{I,i} \phi_{I,i}.$$

in hierarchical decomposition of  $u$

- Use that supports of  $\phi_{I,i}$  are pairwise disjoint
- Maximum-norm

$$\|w_I\|_\infty \leq \frac{h_I^2}{2} \cdot \mu_\infty(u),$$

- $L^2$ -norm

$$\|w_I\|_2^2 = \sum_{i \in I_I} |v_{I,i}|^2 \cdot \|\phi_{I,i}\|_2^2 \leq \frac{h_I^3}{6} \cdot \frac{2h_I}{3} \cdot \sum_{i \in I_I} \mu_2(u|_T)^2 = \frac{h_I^4}{9} \mu_2(u)^2,$$

$$\Rightarrow \|w_I\|_2 \in \mathcal{O}(h_I^2)$$

## Estimation of $w_l$ (2)

- Energy-norm

$$\begin{aligned}\|w_l\|_E^2 &= \sum_{i \in I_l} |v_{l,i}|^2 \cdot \|\phi_{l,i}\|_E^2 = \sum_{i \in I_l} |v_{l,i}|^2 \frac{2}{h_l} \\ &\leq \frac{2}{h_l} \cdot \frac{h_l^4}{4} \cdot \frac{1}{2h_l} \mu_\infty(u)^2 = \frac{h_l^2}{4} \mu_\infty(u)^2\end{aligned}$$

( $2^{l-1} = 1/(2h_l)$  summands)

$\Rightarrow \|w_l\|_E \in \mathcal{O}(h_l)$

## Estimation of $w_l$ (3)

- We can write  $u$  (two times differentiable) as infinite series

$$u = \sum_{l=1}^{\infty} w_l$$

- Convergent in all three norms
- With

$$u - u_n := u - \sum_{l=1}^n w_l = \sum_{l=n+1}^{\infty} w_l$$

in maximum- and  $L^2$ -norm  $\mathcal{O}(h_n^2)$ , in energy-norm  $\mathcal{O}(h_n)$