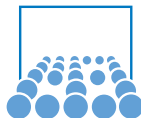


# Algorithms of Scientific Computing

## Space-Filling Curves and their Applications in Scientific Computing III

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Summer Term 2010





# Properties of the Approximating Polygon

- the approximating Polygon connects the **corners** of the recursively divided subsquares
- the connected corners are start and end points of the space-filling curve within each subsquare
  - ⇒ **assists in the construction of space-filling curves**
- approximating polygons are constructed by recursive repetition of a so-called *Leitmotiv*
  - ⇒ **similarity to Koch and other fractal curves**
- the sequence of corresponding functions  $p_n(t)$  converges *uniformly* towards  $h$ 
  - ⇒ additional proof of continuity of the Hilbert curve



# Arithmetic Formulation of the Peano Function

$t$  given in “nonal” system,  $t = 0_9.n_1n_2n_3n_4\dots$ , then

$$p(0_9.n_1n_2n_3n_4\dots) = P_{n_1} \circ P_{n_2} \circ P_{n_3} \circ P_{n_4} \circ \dots \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with the operators

$$\begin{array}{lll}
 P_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{3}x + 0 \\ \frac{1}{3}y + \frac{2}{3} \end{pmatrix} & P_3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{3}x + \frac{1}{3} \\ -\frac{1}{3}y + 1 \end{pmatrix} & P_8 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{3}x + \frac{2}{3} \\ \frac{1}{3}y + \frac{2}{3} \end{pmatrix} \\
 P_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{3}x + \frac{1}{3} \\ \frac{1}{3}y + \frac{1}{3} \end{pmatrix} & P_4 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{3}x + \frac{2}{3} \\ -\frac{1}{3}y + \frac{2}{3} \end{pmatrix} & P_7 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{3}x + 1 \\ \frac{1}{3}y + \frac{1}{3} \end{pmatrix} \\
 P_0 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{3}x + 0 \\ \frac{1}{3}y + 0 \end{pmatrix} & P_5 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{3}x + \frac{1}{3} \\ -\frac{1}{3}y + \frac{1}{3} \end{pmatrix} & P_6 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{3}x + \frac{2}{3} \\ \frac{1}{3}y \end{pmatrix}
 \end{array}$$

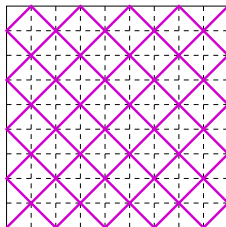
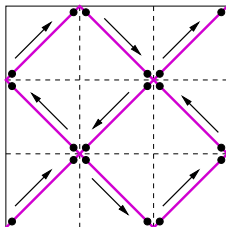
# Approximating Polygons of the Peano Curve

## Definition:

The straight connection between the  $9^n + 1$  points

$$p(0), p(1 \cdot 9^{-n}), p(2 \cdot 9^{-n}), \dots, p((9^n - 1) \cdot 9^{-n}), p(1)$$

is called *n-th approximating polygon of the Peano curve*



# Peano's Representation of the Peano Curve

**Definition:** (Peano curve, original construction by G. Peano)

- each  $t \in \mathcal{I} := [0, 1]$  has a ternary representation

$$t = (0_3.t_1 t_2 t_3 t_4 \dots)$$

- define the mapping  $p: \mathcal{I} \rightarrow \mathcal{Q} := [0, 1] \times [0, 1]$  as

$$p(t) := \begin{pmatrix} 0_3.t_1 k^{t_2}(t_3) k^{t_2+t_4}(t_5) \dots \\ 0_3.k^{t_1}(t_2) k^{t_1+t_3}(t_4) \dots \end{pmatrix}$$

where  $k(t_i) := 2 - t_i$  for  $t_i = 0, 1, 2$  and  $k^j$  is the  $j$ -times concatenation of the function  $k$ .

## Peano's Representation of the Peano Curve (2)

### Still to prove:

- $p$  is independent of the ternary representation
- the Peano curve  $p : \mathcal{I} \rightarrow \mathcal{Q}$  defines a space-filling curve.

### Comments:

- the direction of “switchback” can be both vertical (see definition), horizontal, or mixed;
- actually, *272 different* Peano curves of the switchback type can be constructed using the same principles.  
For comparison: there are only two different 2D Hilbert curves;
- in addition: 2 Peano-Meander curves



# How Long are Approximating Polygons?

## Example: Hilbert curve

- polygon results from recursive repetition of the Leitmotiv
- every recursion step **doubles** the length of the polygon in each subsquare

⇒ length of the  $n$ -th polygon is  $2^n \rightarrow \infty$  for  $n \rightarrow \infty$ .

## Corollaries:

- the “length” of the Hilbert curve is not well defined
- instead, we can give an “area” of the Hilbert curve (1, the area of the unit square)

⇒ **Question: what's the dimension of a Hilbert curve?**

# Fractal Dimension of Curves

Measuring the length of a curve:

- approx. the curve by a polygon with faces of length  $\epsilon$   
 $\Rightarrow$  gives a measured length  $L(\epsilon)$ .  
(cmp. approximating polygons of a space-filling curve)
- in case of recursive repeat of a Leitmotiv:  
replace each units of length  $r$  by a polygon of length  $q$ , then

$$L\left(\frac{\epsilon}{r}\right) = \frac{q}{r}L(\epsilon), \quad L(1) := \lambda$$

- we obtain for the length  $L(\epsilon)$ :

$$L(\epsilon) = \lambda\epsilon^{1-D}, \quad \text{where } D = \log_r q = \frac{\log q}{\log r}$$

## Fractal Dimension of Curves (2)

Length of a recursively defined curve computed as

$$L(\epsilon) = \lambda \epsilon^{1-D}, \quad \text{mit } D = \log_r q = \frac{\log q}{\log r}$$

⇒  $D$  is the **fractal dimension** of the curve

⇒  $\lambda$  is the length w.r.t. that dimension

Gives “well defined” dimension:

- in all other “dimensions”, the length is 0 or  $\infty$ !
- the fractal dimension of the 2D Hilbert curve is 2, similar for the Peano curve

→ **Hausdorff dimension**