

Algorithms of Scientific Computing (Algorithmen des Wissenschaftlichen Rechnens)

Orthogonality

We first consider orthogonality of functions $[a, b] \rightarrow \mathbb{R}$ in the two scalar products we already know

- L^2 scalar product:

$$(u, v)_2 := \int_a^b u(x)v(x) dx$$

- “energy scalar product”:

$$(u, v)_a := \int_a^b u'(x)v'(x) dx,$$

We assume that the space of functions under consideration again be well-defined such that $(u, u) > 0$ for $u \neq 0$ is ensured.

- (i) Show that for $g_k : [0, 2\pi] \rightarrow \mathbb{R}, g_k(x) = \sin(kx)$ and $k, j \in \mathbb{N}$ the L_2 scalar product is

$$(g_k, g_j)_2 = \begin{cases} 0 & \text{for } k \neq j, \\ \pi & \text{else.} \end{cases}$$

- (ii) Which functions of the hierarchical basis are orthogonal to each other w.r.t. the L_2 scalar product? What about the energy scalar product?
- (iii) Let V be a vector space with $\dim V = n < \infty$ with scalar product (\cdot, \cdot) and associated norm $\|x\| := \sqrt{(x, x)}$. Also let $\Psi = \{\psi_1, \dots, \psi_n\} \subset V$ a orthonormal system, i.e.

$$(\psi_i, \psi_j) = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{else.} \end{cases}$$

- a) Show that for

$$x = \sum_{i=1}^n \alpha_i \psi_i$$

the following holds:

$$\|x\| = \sqrt{\sum_{i=1}^n \alpha_i^2}.$$

b) Show that Ψ is a linearly independent system!

c) Show for every $x \in V$:

$$x = \sum_{i=1}^n (x, \psi_i) \psi_i.$$