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Solution Worksheet 1
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Algorithms of Scientific Computing (Algorithmen des Wissenschaftlichen Rechnens)

1 Project: Interpolation of the Trajectory of the Asteroid Pallas – Sample Solution

Excercise 1

see Maple-Worksheet `pallas1.mws`.

Excercise 2

According the Hint:

If $x_l = -X_{12-l}$, then $X_6 = -X_{12-6}$ holds. This is possible only if also $X_6 = 0$ holds. Analog we get $X_0 = -X_{12-0} = -X_{12}$. Here the value X_{12} is the declination for the ascension of 360°, which is the same as the declination value for 0° due to the periodicity of the data. Thus, it must also apply that $X_0 = X_{12}$. So we can conclude that $X_0 = 0$ must hold.

With these considerations we can compute the values a_k and b_k, for example with the Maple-Worksheet `pallas1.mws`.

According the coefficients:

We guess that the interpolation of the axis symmetrical data should only need the axis symmetrical basis functions (i.e. all cos functions), while the interpolation of the point symmetrical data will only depends on the point symmetric basis functions (i.e. all sin functions). Hence, in one case all coefficients $b_k = 0$, while in the other all $a_k = 0$.

To show this we can insert the symmetrical constraints into the equation for the interpolation

$$X_l = a_0 + \sum_{k=1}^{5} \left(a_k \cos\left(\frac{\pi k l}{6}\right) + b_k \sin\left(\frac{\pi k l}{6}\right)\right) + a_6 \cos(\pi l). \quad (1)$$

For X_{12-l} it must hold:

$$\begin{aligned}
X_{12-l} &= a_0 + \sum_{k=1}^{5} \left(a_k \cos\left(\frac{\pi k(12-l)}{6}\right) + b_k \sin\left(\frac{\pi k(12-l)}{6}\right)\right) + a_6 \cos(\pi(12-l)) \\
&= a_0 + \sum_{k=1}^{5} \left(a_k \cos\left(2\pi k - \frac{\pi k l}{6}\right) + b_k \sin\left(2\pi k - \frac{\pi k l}{6}\right)\right) + a_6 \cos(12\pi - l\pi) \\
&= a_0 + \sum_{k=1}^{5} \left(a_k \cos\left(\frac{\pi k l}{6}\right) - b_k \sin\left(\frac{\pi k l}{6}\right)\right) + a_6 \cos(\pi l)
\end{aligned}$$

If $X_l = X_{12-l}$, then holds $X_l - X_{12-l} = 0$, i.e.:

$$\begin{aligned}
& a_0 + \sum_{k=1}^{5} \left(a_k \cos\left(\frac{\pi k l}{6}\right) + b_k \sin\left(\frac{\pi k l}{6}\right)\right) + a_6 \cos(\pi l) \\
- & \left(a_0 + \sum_{k=1}^{5} \left(a_k \cos\left(\frac{\pi k l}{6}\right) - b_k \sin\left(\frac{\pi k l}{6}\right)\right) + a_6 \cos(\pi l) \right) = 0.
\end{aligned}$$

Simplified this results in (for all l):

$$\begin{aligned}
& 2 \sum_{k=1}^{5} b_k \sin\left(\frac{\pi k l}{6}\right) = 0 \\
\Rightarrow \quad & b_k = 0 \quad \text{for all } k.
\end{aligned}$$

Remark: To be precise we would need to show that the solution $b_k = 0$ for all k is the only solution (uniqueness). We will skip this step here and note further that we would generally need to show the uniqueness of the trigonometric interpolation, since otherwise the excercise would not have been properly stated at all. The uniqueness follows from the fact that the matrix is invertible (See the matrix from the lecture, which is, however, for the complex DFT).

Analog we get, if $X_l = -X_{12-l}$ which means $X_l + X_{12-l} = 0$:

$$\begin{aligned}
& 2a_0 + 2 \sum_{k=1}^{5} a_k \cos\left(\frac{\pi k l}{6}\right) + 2a_6 \cos(\pi l) = 0 \\
\Rightarrow \quad & a_k = 0 \quad \text{for all } k.
\end{aligned}$$

Excercise 3

See lecture for the real DFT and the Maple-Worksheet `pallas2.mws`.

2 Two-dimensional Cosine Transform

Excercise 4

We can find a term like

$$\widetilde{G}_k = \frac{1}{N}\sum_{n=0}^{N-1} f_n \cos\left(\frac{\pi k\left(n+\frac{1}{2}\right)}{N}\right)$$

in the equation for $\widetilde{F}_{kl}$ after some small conversions:

$$\begin{aligned}\widetilde{F}_{kl} &= \frac{1}{N\cdot M}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1} f_{nm}\cos\left(\frac{\pi k\left(n+\frac{1}{2}\right)}{N}\right)\cos\left(\frac{\pi l\left(m+\frac{1}{2}\right)}{M}\right)\\ &= \frac{1}{M}\sum_{m=0}^{M-1}\underbrace{\frac{1}{N}\sum_{n=0}^{N-1} f_{nm}\cos\left(\frac{\pi k\left(n+\frac{1}{2}\right)}{N}\right)}\cos\left(\frac{\pi l\left(m+\frac{1}{2}\right)}{M}\right).\end{aligned}$$

Actually there are M of these terms, which we will call

$$\widetilde{G}_{km} = \frac{1}{N}\sum_{n=0}^{N-1} f_{nm}\cos\left(\frac{\pi k\left(n+\frac{1}{2}\right)}{N}\right), \qquad m = 0,\ldots,M-1,$$

With these $\widetilde{G}_{km}$ we can set

$$\widetilde{F}_{kl} = \frac{1}{M}\sum_{m=0}^{M-1}\widetilde{G}_{km}\cos\left(\frac{\pi l\left(m+\frac{1}{2}\right)}{M}\right).$$

These sums can again be computed with the given procedure for each $k = 0,\ldots,N-1$.

In total we get the following algorithm:

1. Compute all N values for all $m = 0,\ldots,M-1$:

$$\widetilde{G}_{km} = \frac{1}{N}\sum_{n=0}^{N-1} f_{nm}\cos\left(\frac{\pi k\left(n+\frac{1}{2}\right)}{N}\right).$$

2. Use the results from all $n = 0,\ldots,N-1$ to compute the N values

$$\widetilde{F}_{kl} = \frac{1}{M}\sum_{m=0}^{M-1}\widetilde{G}_{km}\cos\left(\frac{\pi l\left(m+\frac{1}{2}\right)}{M}\right).$$

We can interpret this algorithm as a 1d transformation which is first applied column-wise on the 2d array and afterwards applied row-wise on the result values.

Remark:

1. We didn't use any special properties of the cosines. So we can apply the same method on all 2d transformations of the shape

$$F_{kl} = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f_{nm} \phi_k (n) \psi_l (m)$$

   anwenden.

2. The extention on the 3d case and even higher-dimensional cases can be achieved in an analog way.