Technische Universität München
Lehrstuhl für Informatik V
Dr. T. Neckel, Dr. D. Pflüger

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Solution 11
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Algorithms of Scientific Computing (Algorithmen des Wissenschaftlichen Rechnens)

Arithmetization of Space-Filling Curves – Solution

Excercise 1: Calculation of h for (in)finite fractions

Initially we calculate the decimal places of the numbers and get:

$$\begin{aligned} \frac{1}{8} &= 0_4.02 \\ \frac{1}{3} &= 0_4.11111111\ldots \end{aligned}$$

So, for $h\left(\frac{1}{8}\right)$ we get:

$$\begin{aligned} h\left(\frac{1}{8}\right) &= H_0 \circ H_2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} = H_0 \left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right) \\ &= H_0 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} \end{aligned}$$

The calculation of $h\left(\frac{1}{3}\right)$ turns out to be much more complicated, since we need to find the following limit:

$$h\left(\frac{1}{3}\right) = h(0_4.1111\ldots) = H_1 \circ H_1 \circ \ldots \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \lim_{n\to\infty} H_1^n \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So, we will write the operator H_1 in the following form:

$$H_1 = \underbrace{\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}}_{=:A_1} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{=:v} + \underbrace{\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}}_{=:b_1} = A_1 v + b_1.$$

From this, we get

$$
\begin{aligned}
H_1^2 v &= A_1(A_1 v + b_1) + b_1 = A_1^2 v + A_1 b_1 + b_1 \\
H_1^3 v &= A_1(A_1^2 v + A_1 b_1 + b_1) + b_1 = A_1^3 v + A_1^2 b + A_1 b_1 + b_1 \\
\vdots &= \quad \vdots \\
H_1^n v &= A_1^n v + A_1^{n-1} b_1 + \ldots + A_1 b_1 + b_1
\end{aligned}
$$

For the term $A_1^{n-1}b_1 + \ldots + A_1 b_1 + b_1$ we use a trick, similiar to that used for geometric progressions:

$$
\begin{aligned}
(I - A_1)\left(A_1^{n-1} b_1 + \ldots + A_1 b_1 + b_1\right) &= A_1^{n-1} b_1 + \ldots + A_1 b_1 + b_1 \\
&\quad - A_1^n b_1 - A_1^{n-1} b_1 - \ldots - A_1 b_1 \\
&= b_1 - A_1^n b_1 = (I - A_1^n)\, b_1
\end{aligned}
$$

This leads to

$$
A_1^{n-1} b_1 + \ldots + A_1 b_1 + b_1 = (I - A_1)^{-1} (I - A_1^n)\, b_1.
$$

which renders to

$$
\begin{aligned}
\lim_{n\to\infty} H_1^n &= \lim_{n\to\infty} \left(A_1^n v + A_1^{n-1} b_1 + \ldots + A_1 b_1 + b_1\right) \\
&= \lim_{n\to\infty} \left((I - A_1)^{-1} (I - \underbrace{A_1^n}_{\to 0}) b_1\right) = (I - A_1)^{-1} b_1
\end{aligned}
$$

I.e. we get the value $h\left(\frac{1}{3}\right) =: \begin{pmatrix} \xi \\ \zeta \end{pmatrix}$ by solving the system of equations

$$
(I - A_1)\begin{pmatrix} \xi \\ \zeta \end{pmatrix} = b_1 \qquad \Leftrightarrow \qquad \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \xi \\ \zeta \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}
$$

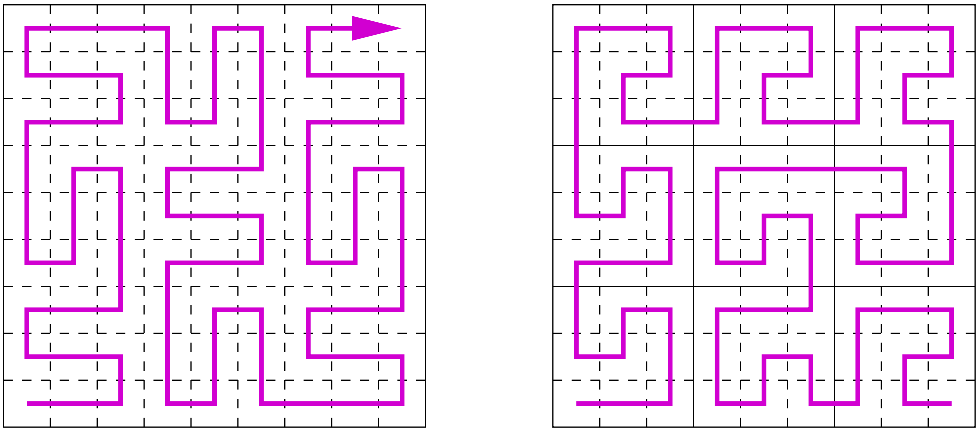
The solution $\xi = 0$ und $\zeta = 1$ can be found quite easily, so finally we get $h\left(\frac{1}{3}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Excercise 2: Arithmetization of the Peano Curve

Like for the arithmetization of the Hilbert curve, we assume that the parameter t is given on the basis 9. $t = 0_9.n_1 n_2 n_3 n_4 \ldots$. Now we are looking for the operators $P_0, \ldots, P_8$, so that

$$
p(0_9.n_1 n_2 n_3 n_4 \ldots) = P_{n_1} \circ P_{n_2} \circ P_{n_3} \circ P_{n_4} \circ \cdots \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$

For the operators P_i and the algorithm for calculating the Peano functions refer to the Python-script `solution11.py`.



(a) Peano curve　　(b) meander-style Peano curve

Abbildung 1: Two space-filling curves of the Peano type