

Algorithms of Scientific Computing (Algorithmen des Wissenschaftlichen Rechnens)

1 Project: Interpolation of the Trajectory of the Asteroid Pallas¹ (Continued)

Exercise 1

$$\begin{aligned} X_l &= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} c_k e^{i2\pi kl/N} = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} c_k \cdot \left(\cos\left(\frac{2\pi kl}{N}\right) + i \sin\left(\frac{2\pi kl}{N}\right) \right) \\ &= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \Re\{c_k\} \cdot \left(\cos\left(\frac{2\pi kl}{N}\right) + i \sin\left(\frac{2\pi kl}{N}\right) \right) + i \Im\{c_k\} \cdot \left(\cos\left(\frac{2\pi kl}{N}\right) + i \sin\left(\frac{2\pi kl}{N}\right) \right) \end{aligned}$$

Now we sort for real and imaginary part of X_l :

$$\begin{aligned} \Re\{X_l\} &= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \Re\{c_k\} \cdot \cos\left(\frac{2\pi kl}{N}\right) - \Im\{c_k\} \cdot \sin\left(\frac{2\pi kl}{N}\right) \\ \Im\{X_l\} &= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \Im\{c_k\} \cdot \cos\left(\frac{2\pi kl}{N}\right) + \Re\{c_k\} \cdot \sin\left(\frac{2\pi kl}{N}\right) \end{aligned}$$

Since X_l is real $\Im\{X_l\}$ must be zero. Thus, the remaining part is

¹Project idea and data are taken from: W. L. Briggs, Van Emden Henson, *The DFT – An Owner's Manual for the Discrete Fourier Transform*, SIAM, 1995

$$X_l = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \Re\{c_k\} \cdot \cos\left(\frac{2\pi kl}{N}\right) - \Im\{c_k\} \cdot \sin\left(\frac{2\pi kl}{N}\right)$$

By reducing the sum to the range $k = 1, \dots, \frac{N}{2} - 1$, we get

$$X_l = \Re\{c_0\} + \Re\{c_{N/2}\} \cdot \cos(\pi l) + \sum_{k=1}^{\frac{N}{2}-1} \Re\{c_k\} \cdot \cos\left(\frac{2\pi kl}{N}\right) + \Re\{c_{-k}\} \cdot \cos\left(\frac{2\pi(-k)l}{N}\right) \\ - \Im\{c_k\} \cdot \sin\left(\frac{2\pi kl}{N}\right) - \Im\{c_{-k}\} \cdot \sin\left(\frac{2\pi(-k)l}{N}\right)$$

Using the symmetry of the sine and cosine functions, this can be derived to

$$X_l = \Re\{c_0\} + \Re\{c_{N/2}\} \cdot \cos(\pi l) + \sum_{k=1}^{\frac{N}{2}-1} (\Re\{c_k\} + \Re\{c_{-k}\}) \cdot \cos\left(\frac{2\pi kl}{N}\right) \\ + (\Im\{c_{-k}\} - \Im\{c_k\}) \cdot \sin\left(\frac{2\pi kl}{N}\right)$$

Since $c_{-k} = c_k^*$ it is $\Re\{c_{-k}\} = \Re\{c_k\}$ and $\Im\{c_{-k}\} = -\Im\{c_k\}$. So, we get

$$X_l = \Re\{c_0\} + \Re\{c_{N/2}\} \cdot \cos(\pi l) + \sum_{k=1}^{\frac{N}{2}-1} 2\Re\{c_k\} \cdot \cos\left(\frac{2\pi kl}{N}\right) - 2\Im\{c_k\} \cdot \sin\left(\frac{2\pi kl}{N}\right)$$

For $N = 12$, $a_k = 2\Re\{c_k\}$, $b_k = -2\Im\{c_k\}$ for all $k = 1, \dots, \frac{N}{2}$, $a_0 = c_0$ and $a_{\frac{N}{2}} = c_{\frac{N}{2}}$ we get equation (2):

$$X_l = a_0 + \sum_{k=1}^5 \left(a_k \cos\left(\frac{\pi kl}{6}\right) + b_k \sin\left(\frac{\pi kl}{6}\right) \right) + a_6 \cos(\pi l)$$

Additional Exercises

Exercise 2

Let ω_N be $\omega_N := e^{i2\pi/N}$.

ω_N^k is the solution of the equation $z^N = 1$ (for all $k \in \mathbb{Z}$)

Assume $z := \omega_N^k$, thus:

$$z^N = (\omega_N^k)^N = \left(e^{\frac{i2\pi k}{N}}\right)^N = e^{\frac{i2\pi Nk}{N}} = e^{i2\pi k} = \underbrace{\cos(2\pi k)}_{=1 \text{ for all } k \in \mathbb{Z}} + i \underbrace{\sin(2\pi k)}_{=0 \text{ for all } k \in \mathbb{Z}} = 1$$

The complex conjugates hold $(\omega_N^k)^* = \omega_N^{-k}$

$$\begin{aligned} (\omega_N^k)^* &= \left(e^{\frac{i2\pi k}{N}}\right)^* = \left(\cos\left(\frac{2\pi k}{N}\right) + i \sin\left(\frac{2\pi k}{N}\right)\right)^* = \left(\cos\left(\frac{2\pi k}{N}\right) - i \sin\left(\frac{2\pi k}{N}\right)\right) \\ &= \left(\cos\left(\frac{2\pi(-k)}{N}\right) + i \sin\left(\frac{2\pi(-k)}{N}\right)\right) = e^{\frac{i2\pi(-k)}{N}} = \omega_N^{-k} \end{aligned}$$

Periodicity

$$\left\{\omega_N^k\right\}_{k=-\infty}^{\infty} = \left\{e^{\frac{i2\pi k}{N}}\right\}_{k=-\infty}^{\infty} = \left\{\cos\left(\frac{2\pi k}{N}\right) + i \sin\left(\frac{2\pi k}{N}\right)\right\}_{k=-\infty}^{\infty}$$

Since both, $\cos\left(\frac{2\pi k}{N}\right)$ and $\sin\left(\frac{2\pi k}{N}\right)$ are N-periodic in $k = -\infty \dots \infty$, also the sum of them must be N-periodic.

Assume a fixed n for the second case. Then

$$\begin{aligned} \left\{\omega_N^{nk}\right\}_{k=-\infty}^{\infty} &= \left\{\cos\left(\frac{2\pi nk}{N}\right) + i \sin\left(\frac{2\pi nk}{N}\right)\right\}_{k=-\infty}^{\infty} \\ &= \left\{\cos\left(n \cdot \frac{2\pi k}{N}\right) + i \sin\left(n \cdot \frac{2\pi k}{N}\right)\right\}_{k=-\infty}^{\infty} \end{aligned}$$

If $\cos\left(\frac{2\pi k}{N}\right)$ is N-periodical, $\cos\left(n \cdot \frac{2\pi k}{N}\right)$ is $\frac{N}{n}$ -periodical and, thus also N-periodical, for $n = -N, \dots, -1, 1, \dots, N$.

If $n = 0$ then $\cos\left(n \cdot \frac{2\pi k}{N}\right) = \cos(0) = 1$, which is trivially N-periodical.

If $|n| > N$ then there is a $c \in \mathbb{Z}$, so that $n := n - c \cdot N$ with $|n'| < N$. Then

$$\begin{aligned} \cos\left(n \cdot \frac{2\pi k}{N}\right) &= \cos\left((cN + n') \cdot \frac{2\pi k}{N}\right) = \cos\left(\frac{2\pi k c N}{N} + \frac{2\pi k n'}{N}\right) = \cos\left(2\pi k + \frac{2\pi k n'}{N}\right) \\ &= \cos\left(\frac{2\pi k n'}{N}\right) \end{aligned}$$

which is again N-periodical, as shown above. The same can be shown for a fixed k .

Exercise 3

The proof follows directly from the last exercise:

$$f_{n+N} = \sum_{k=0}^{N-1} F_k \omega_N^{(n+N)k} = \sum_{k=0}^{N-1} F_k \omega_N^{nk} = f_n$$

and

$$F_{k+N} = \frac{1}{N} \sum_{n=0}^{N-1} f_n \omega_N^{-n(k+N)} = \frac{1}{N} \sum_{n=0}^{N-1} f_n \omega_N^{-nk} = F_k$$