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Sheet 5
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Algorithms of Scientific Computing
(Algorithmen des Wissenschaftlichen Rechnens)
Adaptivity, Norms of Functions
Proposed Solution

1 One-dimensional Sparse Grids—An Adaptive Implementation

a)
- two loops: over levels and indices on each level
- create a grid point for each level-index-pair, evaluate the function and insert the grid point into the map (dictionary)
- hierarchize bottom-up: loop over all points on all levels $minLevel, minLevel - 1, \dots, 2$ (in this order) and subtract from each function value the mean value of the left and right neighbors' function values (if one side is boundary just skip this side); use the functions *computeLeftNeighbor* and *computeRightNeighbor* to identify the hierarchical neighbors

b)
- iterate over all grid points in the map
- compute the triangle areas using the keys and the surpluses
- sum everything up and return the result

c)
- implement the private member function *refineRecursively* that checks whether a grid point's surplus is still so large the grid point should be refined
- if not yet in the map, create a grid point, determine its function value and insert it
- hierarchize the point using the hierarchic neighbors' stored function values
- Only after inserting both children make the recursive call for the children! (Do you know why?)

2 Norms of Functions

1a) $f_k(x) := \sin(k\pi x), \quad k \in \mathbb{N}$

- $\|f_k\|_\infty = 1$ — the only thing interesting is that for every $k > 0$ the function actually assumes this maximum.

- Now we need the antiderivative of f_k^2 (look up, ask maple, partial integration,...):

$$\frac{x}{2} - \frac{sin(k\pi x)cos(k\pi x)}{2k\pi},$$

(don't believe it? take the derivative!). The result is

$$\int_0^1 f_k(x)^2\,dx = \frac{1}{2} \qquad \text{and thus} \qquad \|f_k\|_2 = \sqrt{\frac{1}{2}}$$

(independent of k).

- It works similar for the energy norm since the antiderivative of $(f_k')^2(x) = (k\pi\cos(k\pi x))^2$ is

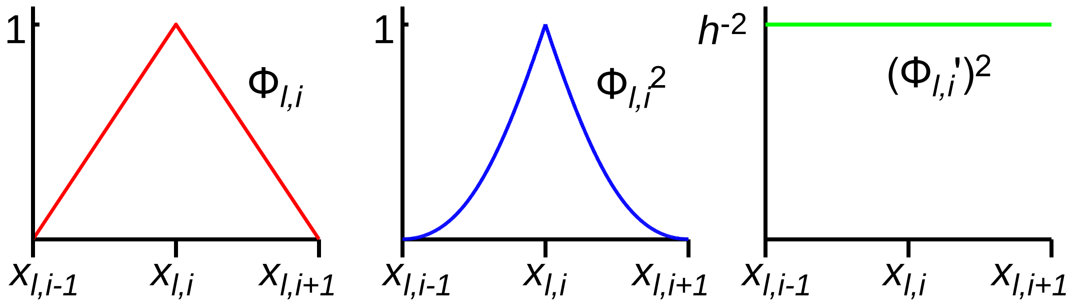
$$(k\pi)^2 \cdot \left(\frac{x}{2} + \frac{sin(k\pi x)cos(k\pi x)}{2k\pi}\right).$$

Apparently in the energy norm

$$\|f_k\|_E = k\pi\sqrt{\frac{1}{2}}.$$

higher frequencies have a stronger influence.

1b) $\phi_{l,i}(x) := \phi\left(2^l x - i\right)$ Integration is easier here, after one has understood what the functions look like (with $x_{l,i} := i \cdot 2^{-l}$ and $h = 2^{-l}$):



- $\|\phi_{l,i}\|_\infty = 1$ (by mere looking) independent of l and i.

- At first we compute the L^2 norm for $\phi = \phi_{0,0}$, i.e. for once we consider $[-1, 1]$ instead of $[0, 1]$. We get

$$\int_{-1}^1 \phi(x)^2\,dx = 2\int_0^1 x^2\,dx = 2\left[\frac{x^3}{3}\right]_{x=0}^1 = \frac{2}{3},$$

and thus $\|\phi\|_2 = \sqrt{2/3}$ (still in $[-1, 1]$!).

In order to transform ϕ to $\phi_{l,i}$ we translate it (no change to the integral) and scale it by a factor $h = 2^{-l}$. Taking the scaling into consideration the norm can be rewritten as

$$\|\phi_{l,i}\|_2 = \sqrt{\frac{2}{3}2^{-l}}.$$

"Small looking" basis functions apparently are also less important.

- The picture on the right allows us to directly determine the energy norm (the quadrangle has width $2 \cdot 2^{-l}$ and height 2^{2l}):

$$\|\phi_{l,i}\|_E = \sqrt{2 \cdot 2^l}.$$

  Here, "small looking" basis functions are *more* important!

2) For each of these norms prove the *triangle inequality*

$$\|u + v\| \leq \|u\| + \|v\|.$$

For the L^2 norm use the Cauchy-Schwarz inequality

$$|(u, v)| \leq \|u\| \cdot \|v\|,$$

that holds for arbitrary scalar products, i.e. also for the L^2 scalar product.

- Infinity norm: Let $x \in [0, 1]$ such that $\|u + v\|_\infty = |u(x) + v(x)|$. We directly get

$$\|u + v\|_\infty = |u(x) + v(x)| \leq |u(x)| + |v(x)| \leq \|u\|_\infty + \|v\|_\infty$$

  (for both "$\leq$" the case of "$<$" is possible, but that's not important).

- L^2 norm:

$$\begin{aligned} \|u + v\|_2^2 &= (u + v, u + v)_2 \\ &= (u, u)_2 + (u, v)_2 + (v, u)_2 + (v, v)_2 \\ &\leq (u, u)_2 + 2|(u, v)_2| + (v, v)_2 \\ &\leq (u, u)_2 + 2\|u\|_2\|v\|_2 + (v, v)_2 \\ &= (\|u\| + \|v\|)^2. \end{aligned}$$

- Energy norm: see definition and previous item