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SS 2011
Sheet 8
7.7.2011

Algorithms of Scientific Computing (Algorithmen des Wissenschaftlichen Rechnens)

Orthogonality

Proposed solution

We first consider orthogonality of functions $[a, b] \to \mathbb{R}$ in the two scalar products we already know

- L^2 scalar product:
$$(u, v)_2 := \int_a^b u(x)v(x)\; dx$$
- "energy scalar product":
$$(u, v)_a := \int_a^b u'(x)v'(x)\; dx,$$

We assume that the space of functions under consideration again be well-defined such that $(u, u) > 0$ for $u \neq 0$ is ensured.

(i) Show that for $g_k : [0, 2\pi] \to \mathbb{R}, g_k(x) = \sin(kx)$ and $k, j \in \mathbb{N}$ the L_2 scalar product is

$$(g_k, g_j)_2 = \begin{cases} 0 & \text{for } k \neq j, \\ \pi & \text{else.} \end{cases}$$

$$\begin{aligned}\int_0^{2\pi} \sin(kx)\sin(jx)\; dx &= \underbrace{-\frac{1}{k}\cos(kx)\sin(jx)\Big|_0^{2\pi}}_{=0} + \frac{j}{k}\int_0^{2\pi}\cos(kx)\cos(jx)\; dx \\ &= \underbrace{\frac{j}{k^2}\sin(kx)\cos(jx)\Big|_0^{2\pi}}_{=0} + \frac{j^2}{k^2}\int_0^{2\pi}\sin(kx)\sin(jx)\; dx\end{aligned}$$

This only holds for $j \neq k$ iff $(g_k, g_j)_2 = 0$

Recalling the results from worksheet 5 (back then the integral's domain was $[0, 1]$) we know $(g_k, g_k) = \|g_k\|_2^2 = \pi$.

(ii) Which functions of the hierarchical basis are orthogonal to each other w.r.t. the L_2 scalar product? What about the energy scalar product?
Because of $\phi_{l,i}(x) \geq 0$ we have $(\phi_{l,i}, \phi_{l',j})_2 = 0$ iff the supports are disjoint (i.e. there's no ancestor relation between them in the binary tree).

For arbitrary pairs however we get for the energy scalar product $(\phi_{l,i}, \phi_{l',j})_a = 0$ (draw derivatives and think about how basis functions influence each other).
A direct implication of this is that in the one-dimensional case the stiffness matrix containing the energy scalar product of the hierarchical basis functions is a diagonal matrix!

(iii) Let V be a vector space with $\dim V = n < \infty$ with scalar product $(\,.\,,\,.\,)$ and associated norm $\|x\| := \sqrt{(x,x)}$. Also let $\Psi = \{\psi_1, \ldots, \psi_n\} \subset V$ a orthonormal system, i.e.

$$(\psi_i, \psi_j) = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{else.} \end{cases}$$

a) Show that for

$$x = \sum_{i=1}^{n} \alpha_i \psi_i$$

the following holds:

$$\|x\| = \sqrt{\sum_{i=1}^{n} \alpha_i^2}.$$

$$\|x\|^2 = (x,x) = \left(\sum_{i=1}^{n} \alpha_i \psi_i, \sum_{j=1}^{n} \alpha_j \psi_j \right) = \sum_{i,j=1}^{n} (\alpha_i \psi_i, \alpha_j \psi_j) = \sum_{i=1}^{n} \alpha_i^2$$

Now **assume** *for a moment that the hierarchical basis was an orthonormal basis: What would the error estimation for the error $\|u - u_L\|$ look like?*

$$\|u - u_L\| \leq \sum_{\vec{l} \notin L} \|w_{\vec{l}}\|$$

could be rewritten as an exact equation

$$\|u - u_L\| = \sum_{\vec{l} \notin L} \sum_{\vec{i} \in I_{\vec{l}}} (v_{\vec{l},\vec{i}})^2$$

b) Show that Ψ is a linearly independent system!

If $0 = \sum \alpha_i \psi_i$ then with the previous item we get $\alpha_1 = \ldots = \alpha_n = 0$.

c) Show for every $x \in V$:

$$x = \sum_{i=1}^{n} (x, \psi_i) \psi_i.$$

Since we have n linearly independent ψ_i there's always a well-defined set of unique α_i with

$$x = \sum_{i=1}^{n} \alpha_i \psi_i.$$

Therefore we have

$$(x, \psi_j) = \left(\sum_{i=1}^{n} \alpha_i \psi_i, \psi_j \right) = \alpha_j.$$