Technische Universität München
Lehrstuhl für Informatik V
Dr. T. Neckel, Dr. D. Pflüger

Algorithms of Scientific Computing (Algorithmen des Wissenschaftlichen Rechnens)

Notes about the integral representation of the hierarchical surplus

Theorem:

It holds

$$v_{l,i} := u(x_{l,i}) - \frac{1}{2}(u(x_{l,i-1}) + u(x_{l,i+1})) = \int_\Omega \psi_{l,i}(x)u''(x)dx; \qquad \psi_{l,i}(x) = -\frac{h_l}{2}\phi_{l,i}(x)$$

Proof:

$$\begin{aligned} v_{l,i} &= \int_\Omega -\frac{h_l}{2}\phi_{l,i}(x)u''(x)dx \\ &= -\frac{h_l}{2}(\underbrace{\int_{x_{l,i-1}}^{x_{l,i}} \phi_{l,i}(x)u''(x)dx}_{I} + \underbrace{\int_{x_{l,i}}^{x_{l,i+1}} \phi_{l,i}(x)u''(x)dx)}_{II} \end{aligned}$$

$$\begin{aligned} I &= [\phi_{l,i}(x)u'(x)]_{x_{l,i-1}}^{x_{l,i}} - \int_{x_{l,i-1}}^{x_{l,i}} \phi'_{l,i}(x)u'(x)dx && \text{partial integration} \\ &= u'(x_{l,i}) - [\phi'_{l,i}(x)u(x)]_{x_{l,i-1}}^{x_{l,i}} + \int_{x_{l,i-1}}^{x_{l,i}} \phi''_{l,i}(x)u(x)dx && \text{partial integration} \\ &= u'(x_{l,i}) - h_l^{-1}u(x_{l,i}) + h_l^{-1}u(x_{l,i-1}) \end{aligned}$$

$$II = -u'(x_{l,i}) + h_l^{-1}u(x_{l,i+1}) + h_l^{-1}u(x_{l,i})$$

$$\begin{aligned} v_{l,i} &= -\frac{h_l}{2}(I + II) \\ &= -\frac{h_l}{2}\left(-2h_l^{-1}u(x_{l,i}) + h_l^{-1}u(x_{l,i-1}) + h_l^{-1}u(x_{l,i+1})\right) \\ &= u(x_{l,i}) - \frac{1}{2}(u(x_{l,i-1}) + u(x_{l,i+1})) \end{aligned}$$

□