Algorithms of Scientific Computing

Hierarchical Methods and Sparse Grids

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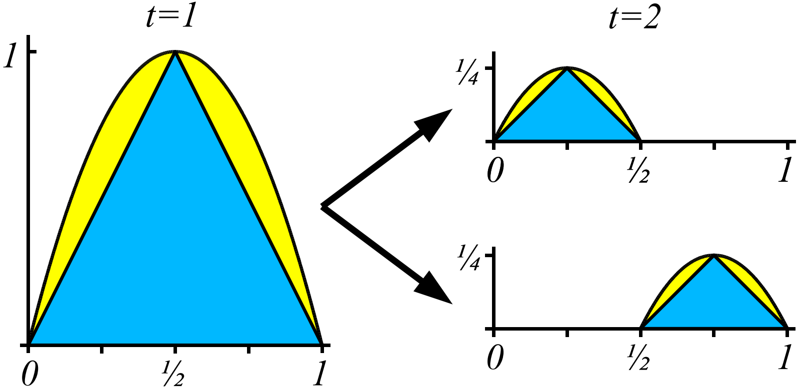
Part III

Hierarchical Decomposition, $1d$

Archimedes' Quadrature

Compute an approximation of $F_1 := \int_0^1 4 \cdot x \cdot (1 - x)\, dx = \frac{2}{3}$



Archimedes' Quadrature (2)

- Integrating $4x(1-x)$, we have to consider several quantities
- Ordered by (recursive) level t:

Level-depth	1	2	3	4	...	t
Mesh-width h	1/2	1/4	1/8	1/16	...	2^{-t}
# triangles	1	2	4	8	...	$\frac{1}{2}2^{t}$
surplus v	1	1/4	1/16	1/64	...	$4 \cdot 2^{-2t}$
Area of triangle D_1	1/2	1/16	1/128	1/1024	...	$4 \cdot 2^{-3t}$
Sum (current t)	1/2	1/8	1/32	1/128	...	$2 \cdot 2^{-2t}$
Sum ($\leq t$)	1/2	5/8	21/32	85/128	...	$\frac{2}{3}\left(1-2^{-2t}\right)$
Error	1/6	1/24	1/96	1/384	...	$\frac{2}{3}2^{-2t}$

Approximation of Functions

- To analyze Archimedes' quadrature rule, we consider functions
- We need a representation of the (approximating) function $u(x)$ which we are integrating:
  - u as linear combination of ansatz functions ϕ_i:

$$u(x) = \sum_{i=1}^{n} \alpha_i \cdot \phi_i(x)$$

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$$u(x) = \sum_{i=1}^{n} \alpha_i \cdot \phi_i(x)$$

- Integrating $u(x)$:

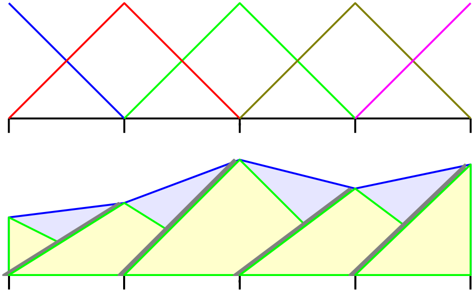
$$\int_a^b u(x)\,dx = \sum_i^n \alpha_i \int_a^b \phi_i(x)\,dx,$$

  - Weighted sum of α_i
  - Remember: Newton-Cotes formulas are weighted sum of function evaluations

Composite Trapezoidal Rule: Function

Interpolant

- Continuous, piecewise linear function
- Represent u in nodal point (hat) basis



- Koefficients α_i are function values at grid points
- Ansatz functions have area h ($h/2$ at boundaries)

Piecewise Linear Functions

Ansatz space

- Only consider $u : [0, 1] \to \mathbb{R}$
- Consider discretization level $n \in \mathbb{N}$
- Obtain
  - *Mesh-width* $h_n = 2^{-n}$
  - *Grid points* $x_{n,i} = i \cdot h_n$

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- Obtain
  - *Mesh-width* $h_n = 2^{-n}$
  - *Grid points* $x_{n,i} = i \cdot h_n$
  - Define "mother of all hat functions"

$$\phi(x) := \max\{1 - |x|, 0\}$$

$\Rightarrow$ *Ansatz functions*

$$\phi_{n,i}(x) := \phi\left(\frac{x - x_{n,i}}{h_n}\right)$$

- Nodal point basis $\Phi_n := \{\phi_{n,i}, 0 \leq i \leq 2^n\}$

Piecewise Linear Functions (2)

- Space of continous piecewise linear functions

$$V_n = \operatorname{span}(\Phi_n)$$

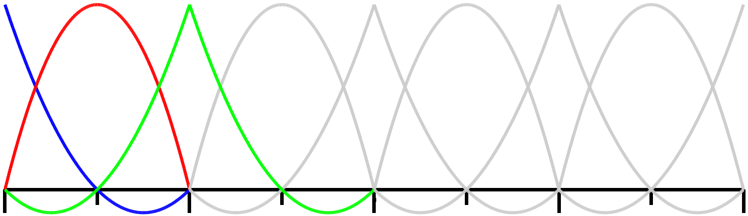
- Interpolants $u_n \in V_n$

$$u_n(x) = \sum_{i=0}^{2^n} \alpha_{n,i}\phi_{n,i}(x)$$

Composite Simpson's Rule: Function

Interpolant

- Continuous, piecewise quadratic function
- More complicated basis:



- Ansatz functions: Lagrangian polynomials, glued together
- α_i: function values at grid points
- Ansatz functions have area $h/6$ (blue), $4h/6$ (red), $2h/6$ (green)
- We'll not formally define basis functions here…

From Composite Trapezoidal to Archimedes

Piecewise linear functions

- We restrict our functions u to $u(0) = u(1) = 0$
- Nodal point basis for discretization level n:

$$\Phi_n := \{\phi_{n,i}, 1 \leq i \leq 2^n - 1\}$$

- Function space

$$V := \bigcup_{l=1}^{\infty} V_l$$

  contains all functions which are for sufficiently large l in V_l
- Generating system of V as

$$\Phi := \bigcup_{l=1}^{\infty} \Phi_l$$

  - Note: not minimal, thus not a basis (not linear independent)

Hierarchical Basis

- We are interested in a hierarchical decomposition of V_l
  - Define *hierarchical increment* W_l, s.t. V_l is a *direct sum* of W_l:

$$V_l = V_{l-1} \oplus W_l$$

Side-note: direct sum

- Every $u_l \in V_l$ can be uniquely decomposed as $u_l = u_{l-1} + w_l$, with $u_{l-1} \in V_{l-1}$ and $w_l \in W_l$

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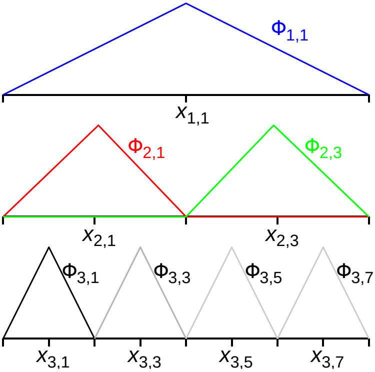
- W_l has to contain 2^{l-1} ansatz functions: $\dim V_l = 2^l - 1 = \dim V_{l-1} + \dim W_l$
- This holds (introducing index sets I_l) for

$$I_l := \{i : 1 \leq i < 2^l,\ i \text{ odd}\}$$

$$W_l := \text{span}\,\{\phi_{l,i} :\ i \in I_l\}$$

Hierarchical Increments

- Set of hierarchical increments W_l
- For $l = 1$: $W_1 = V_1$
- Example for $l = 1, 2, 3$:



Hierarchical Basis (cont.)

- Then

$$V_n = \bigoplus_{l=1}^{n} W_l$$

is a direct sum, too:

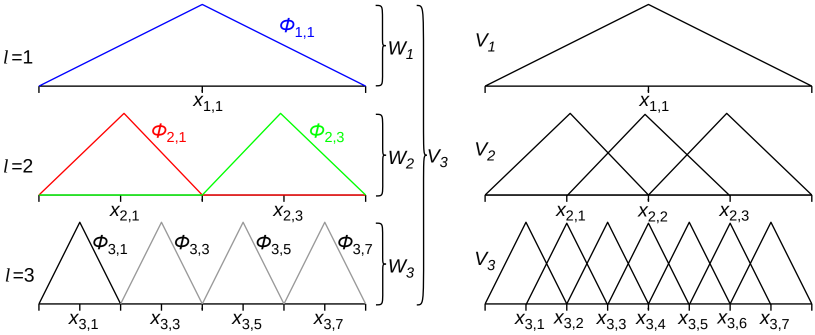
- $u \in V_n$ can be decomposed uniquely into $w_l \in W_l$:

$$u = \sum_{l=1}^{n} w_l = \sum_{l=1}^{n} \sum_{i \in I_l} v_{l,i} \phi_{l,i}$$

  - Coefficients $v_{l,i}$ are hierarchical surplusses
- Corresponding basis of V_n (or, with ∞ instead of n, of V)

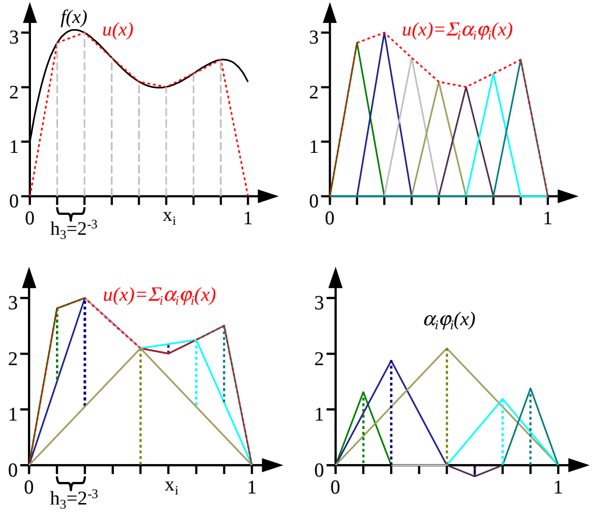
$$\Psi_n := \bigcup_{l=1}^{n} \{\phi_{l,i} : i \in I_l\}.$$

Comparison



$l=1$
$\Phi_{1,1}$
W_1
$x_{1,1}$
$l=2$
$\Phi_{2,1}$
$\Phi_{2,3}$
W_2
V_3
$x_{2,1}$
$x_{2,3}$
$l=3$
$\Phi_{3,1}$
$\Phi_{3,3}$
$\Phi_{3,5}$
$\Phi_{3,7}$
W_3
$x_{3,1}$ $x_{3,3}$ $x_{3,5}$ $x_{3,7}$
V_1
$x_{1,1}$
V_2
$x_{2,1}$ $x_{2,2}$ $x_{2,3}$
V_3
$x_{3,1}$ $x_{3,2}$ $x_{3,3}$ $x_{3,4}$ $x_{3,5}$ $x_{3,6}$ $x_{3,7}$

Comparison (2)



Analysis of Hierarchical Decomposition

- Contribution of summands in hierarchical decomposition

$$u = \sum_{l=1}^{n} w_l = \sum_{l=1}^{n} \sum_{i \in I_l} v_{l,i} \phi_{l,i}.$$

- Interesting in univariate setting
- Will be crucial in mulitvariate setting
  - Cost/benefit analysis will help to significantly reduce effort
- Need several norms to measure w_l (cf. worksheet 5)

Norms of Functions

Again, we assume sufficiently smooth functions $u : [0, 1] \to \mathbb{R}$

Norms

- Maximum-norm

$$\|u\|_\infty := \max_{x \in [0,1]} |u(x)|$$

- L^2-norm

$$\|u\|_2 := \sqrt{\int_0^1 u(x)^2 \, dx},$$

  for the L^2 scalar product

$$(u, v)_2 := \int_0^1 u(x)v(x) \, dx$$

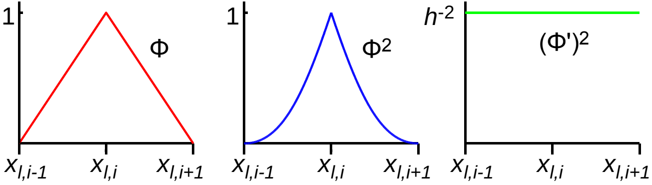
- Energy-norm

$$\|u\|_E := \|u'\|_2$$

Norms of Basis Functions

For the basis functions $\phi_{l,i}$, we obtain

$$
\begin{aligned}
\|\phi_{l,i}\|_\infty &= 1 \\
\|\phi_{l,i}\|_2 &= \sqrt{\frac{2h_l}{3}} \\
\|\phi_{l,i}\|_E &= \sqrt{\frac{2}{h_l}}
\end{aligned}
$$



Estimation of Surplusses

- Let $\psi_{l,i} := -\frac{h_l}{2}\phi_{l,i}$
- Surplus $v_{l,i}$ of basis function $\phi_{l,i}$
- u two times differentiable

⇒ We can then write $v_{l,i}$ as

$$v_{l,i} = \int_0^1 \psi_{l,i}(x)u''(x)\,dx.$$

- $v_{l,i}$ depends on u'', thus we define for future use

$$\mu_2(u) := \|u''\|_2 \quad \textit{und} \quad \mu_\infty(u) := \|u''\|_\infty.$$

Estimation of Surplusses (2)

- Starting from integral representation of $v_{l,i}$, we can bound

$$|v_{l,i}| \leq \frac{h_l}{2} \cdot \left(\int_0^1 \phi_{l,i} \, dx \right) \cdot \mu_\infty(u) = \frac{h_l^2}{2} \cdot \mu_\infty(u)$$

- and (via Cauchy-Schwartz inequality $|(u,v)| \leq \|u\| \cdot \|v\|$)

$$|v_{l,i}| \leq \frac{h_l}{2} \|\phi_{l,i}\|_2 \cdot \mu_2(u|_{T_i}) = \sqrt{\frac{h_l^3}{6}} \cdot \mu_2(u|_{T_i}),$$

- $u|_{T_i}$ restricts u to the support $T_i = [x_{l,i-1}, x_{l,i+1}]$ of $\phi_{l,i}$

Estimation of w_l

- Estimate contribution of

$$w_l = \sum_{i \in I_l} v_{l,i} \phi_{l,i}.$$

  in hierarchical decomposition of u
- Use that supports of $\phi_{l,i}$ are pairwise disjoint
- Maximum-norm

$$\|w_l\|_\infty \leq \frac{h_l^2}{2} \cdot \mu_\infty(u),$$

- L^2-norm

$$\|w_l\|_2^2 = \sum_{i \in I_l} |v_{l,i}|^2 \cdot \|\phi_{l,i}\|_2^2 \leq \frac{h_l^3}{6} \cdot \frac{2h_l}{3} \cdot \sum_{i \in I_l} \mu_2(u|_{T_i})^2 = \frac{h_l^4}{9} \mu_2(u)^2,$$

$$\Rightarrow \|w_l\|_2 \in \mathcal{O}(h_l^2)$$

Estimation of w_l (2)

- Energy-norm

$$
\begin{aligned}
\|w_l\|_E^2 &= \sum_{i \in I_l} |v_{l,i}|^2 \cdot \|\phi_{l,i}\|_E^2 = \sum_{i \in I_l} |v_{l,i}|^2 \frac{2}{h_l} \\
&\leq \frac{2}{h_l} \cdot \frac{h_l^4}{4} \cdot \frac{1}{2h_l} \mu_\infty(u)^2 = \frac{h_l^2}{4} \mu_\infty(u)^2
\end{aligned}
$$

($2^{l-1} = 1/(2h_l)$ summands)
$\Rightarrow \|w_l\|_E \in \mathcal{O}(h_l)$

Estimation of w_l (3)

- We can write u (two times differentiable) as infinite series

$$u = \sum_{l=1}^{\infty} w_l$$

- Convergent in all three norms
- With

$$u - u_n := u - \sum_{l=1}^{n} w_l = \sum_{l=n+1}^{\infty} w_l$$

in maximum- and L^2-norm $\mathcal{O}(h_n^2)$, in energy-norm $\mathcal{O}(h_n)$