Algorithms of Scientific Computing

Space-Filling Curves and their Applications in Scientific Computing II

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Space-filling Curves – Algorithms

Traversal of h-indexed objects:

- given a set of objects with “positions” $p_i \in \mathcal{Q}$
- traverse all objects, such that $\bar{h}^{-1}(p_{i_0}) < \bar{h}^{-1}(p_{i_1}) < \dots$

Compute mapping:

- for a given index $t \in \mathcal{I}$, compute the image $h(t)$

Compute the index of a given point:

- given $p \in \mathcal{Q}$, find a parameter t, such that $h(t) = p$
- problem: inverse of h is not unique (h not bijective!)
- define a “technically unique” inverse mapping $\bar{h}^{-1}$

Arithmetic Formulation of the Hilbert Curve

Idea:

- interval sequence within the parameter interval $\mathcal{I}$ corresponds to a **quaternary representation**; e.g.:

$$\left[\tfrac{1}{4}, \tfrac{2}{4}\right] = [0_4.1, 0_4.2]\,, \quad \left[\tfrac{3}{4}, 1\right] = [0_4.3, 1_4.0]$$

- **self-similarity**: every subsquare of the target domain contains a scaled, translated, and rotated/reflected Hilbert curve.

$\Rightarrow$ **Construction** of the arithmetic representation:

- find quaternary representation of the parameter
- use quaternary coefficients to determine the required sequence of operations

Arithmetic Formulation of the Hilbert Curve (2)

Revursive approach:

$$h(0_4.q_1 q_2 q_3 q_4 \ldots) = H_{q_1} \circ h(0_4.q_2 q_3 q_4 \ldots)$$

- $\tilde{t} = 0_4.q_2 q_3 q_4 \ldots$ is the relative parameter in the subinterval $[0_4.q_1, 0_4.(q_1 + 1)]$
- $h(\tilde{t}) = h(0_4.q_2 q_3 q_4 \ldots)$ is the relative position of the curve point in the subsquare
- H_{q_1} transforms $h(\tilde{t})$ to its correct position in the unit square:
  - rotation
  - translation
- expanding the recursion equation leads to:

$$h(0_4.q_1 q_2 q_3 q_4 \ldots) = H_{q_1} \circ H_{q_2} \circ H_{q_3} \circ H_{q_4} \circ \cdots$$

Arithmetic Formulation of the Hilbert Curve (3)

If t is given in quaternary digits, i.e. $t = 0_4.q_1 q_2 q_3 q_4 \ldots$, then $h(t)$ may be represented as

$$h(0_4.q_1 q_2 q_3 q_4 \ldots) = H_{q_1} \circ H_{q_2} \circ H_{q_3} \circ H_{q_4} \circ \cdots \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

using the following operators:

$$H_0 := \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} \frac{1}{2}y \\ \frac{1}{2}x \end{pmatrix} \qquad H_1 := \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} \frac{1}{2}x \\ \frac{1}{2}y + \frac{1}{2} \end{pmatrix}$$

$$H_2 := \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} \frac{1}{2}x + \frac{1}{2} \\ \frac{1}{2}y + \frac{1}{2} \end{pmatrix} \qquad H_3 := \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} -\frac{1}{2}y + 1 \\ -\frac{1}{2}x + \frac{1}{2} \end{pmatrix}$$

Matrix Form of the Operators $H_0, \ldots, H_3$

In matrix notation, the operators $H_0, \ldots, H_3$ are:

$$H_0 := \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad H_1 := \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$$

$$H_2 := \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \qquad H_3 := \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}$$

Governing operations:

- scale with factor $\frac{1}{2}$
- translate start of the curve, e.g. $+\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$
- reflect at x and y axis (for H_3)

A First Comment Concerning Uniqueness

Question:

Are the values $h(t)$ independent of the choice of quaternary representation of t concerning trailing zeros:

$$h(0_4.q_1 \ldots q_n) = h(0_4.q_1 \ldots q_n 000 \ldots),$$

Outline of the proof:

1. compute the limit $\lim_{n\to\infty} H_0^n$, or $\lim_{n\to\infty} H_0^n \begin{pmatrix} x \\ y \end{pmatrix}$;
   Result: $\lim_{n\to\infty} H_0^n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
2. show: $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a fixpoint of H_0, i. e. $H_0 \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$\Rightarrow$ independence of trailing zeros, as H_{q_n} is applied to the fixpoint!

A Second Comment Concerning Uniqueness

Question:

Are the values $h(t)$ independent of the choice of quaternary representation of t, as in:

$$h(0_4.q_1 \dots q_n) = h(0_4.q_1 \dots q_{n-1}(q_n - 1)333\dots), \qquad q_n \neq 0$$

(if $q_n = 0$, then consider $0_4.q_1 \dots q_n = 0_4.q_1 \dots q_{n-1}$)

Outline of the proof:

1. compute the limits $\lim_{n\to\infty} H_0^n$ and $\lim_{n\to\infty} H_3^n$.
2. for $q_n = 1, 2, 3$, show that

$$H_{q_n} \circ \lim_{n\to\infty} H_0^n = H_{q_n-1} \circ \lim_{n\to\infty} H_3^n$$

Algorithm to Compute the Hilbert Mapping

Task: given a parameter t, find $h(t) = (x, y) \in \mathcal{Q}$

Most important subtasks:

1. compute quaternary digits – use multiply by 4:

$$4 \cdot 0_4.q_1 q_2 q_3 q_4 \ldots = (q_1.q_2 q_3 q_4 \ldots)_4$$

   and cut off the integer part

2. apply operators H_q in the correct sequence – use recursion:

$$h(0_4.q_1 q_2 q_3 q_4 \ldots) = H_{q_1} \circ H_{q_2} \circ H_{q_3} \circ H_{q_4} \circ \cdots \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

3. stop recursion, when a given tolerance is reached
   $\Rightarrow$ track size of interval or set number of digits

Computing the Inverse Mapping

Task: find a parameter t, such that $h(t) = (x, y)$ for a given $(x, y) \in \mathcal{Q}$

Problem: h not bijective; hence, t is not unique

$\Rightarrow$ a strict inverse mapping h^{-1} does not exist

$\Rightarrow$ instead, compute a "technically unique" inverse $\bar{h}^{-1}$

Recursive Idea:

- determine the subsquare that contains (x, y)
- transform (using the inverse operations of $H_0, \ldots, H_3$) the point (x, y) into the original domain $\rightarrow (\tilde{x}, \tilde{y})$
- recursively compute a parameter $\tilde{t}$ that is mapped to $(\tilde{x}, \tilde{y})$
- depending on the subsquare, compute t from $\tilde{t}$

Inverse Operators of $H_0, \ldots, H_3$

$$\begin{pmatrix} x \\ y \end{pmatrix} = H_0 \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\tilde{y} \\ \frac{1}{2}\tilde{x} \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} 2y \\ 2x \end{pmatrix}$$

By similar computations:

$$H_0^{-1} := \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2y \\ 2x \end{pmatrix} \qquad H_1^{-1} := \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2x \\ 2y-1 \end{pmatrix}$$

$$H_2^{-1} := \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2x-1 \\ 2y-1 \end{pmatrix} \qquad H_3^{-1} := \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -2y+1 \\ -2x+2 \end{pmatrix}$$

Algorithm to Compute the Inverse Mapping

$\bar{h}^{-1} := \texttt{proc}(x, y)$

(1) determine the subsquare $q \in \{0, \ldots, 3\}$ by checking $x <> \frac{1}{2}$ and $y <> \frac{1}{2}$:

1	2
0	3

(treat cases $x, y = \frac{1}{2}$ in a unique way: either $<$ or $>$ $\Rightarrow$ *technically unique inverse*)

(2) set $(\tilde{x}, \tilde{y}) := H_q^{-1}(x, y)$

(3) recursively compute $\tilde{t} := \bar{h}^{-1}(\tilde{x}, \tilde{y})$

(4) return $t := \frac{1}{4}\left(q + \tilde{t}\right)$ as value

(stopping criterion still to be added)