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Sheet 1
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Algorithms of Scientific Computing

1 Project: Interpolation of the Trajectory of the Asteroid Pallas – Sample Solution

Maple Demo

see Maple-Worksheet `pallas1.mws`.

Excercise 1

$$
\begin{aligned}
X_l &= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} c_k e^{i2\pi kl/N} = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} c_k \cdot \left(\cos\left(\frac{2\pi kl}{N}\right) + i \sin\left(\frac{2\pi kl}{N}\right)\right) \\
&= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \Re\{c_k\} \cdot \left(\cos\left(\frac{2\pi kl}{N}\right) + i \sin\left(\frac{2\pi kl}{N}\right)\right) + i\Im\{c_k\} \cdot \left(\cos\left(\frac{2\pi kl}{N}\right) + i \sin\left(\frac{2\pi kl}{N}\right)\right)
\end{aligned}
$$

Now we sort for real and imaginery part of X_l:

$$
\begin{aligned}
\Re\{X_l\} &= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \Re\{c_k\} \cdot \cos\left(\frac{2\pi kl}{N}\right) - \Im\{x_k\} \cdot \sin\left(\frac{2\pi kl}{N}\right) \\
\Im\{X_l\} &= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \Im\{c_k\} \cdot \cos\left(\frac{2\pi kl}{N}\right) + \Re\{x_k\} \cdot \sin\left(\frac{2\pi kl}{N}\right)
\end{aligned}
$$

Since X_l is real $\Im\{X_l\}$ must be zero. Thus, the remaining part is

$$
X_l = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \Re\{c_k\} \cdot \cos\left(\frac{2\pi kl}{N}\right) - \Im\{x_k\} \cdot \sin\left(\frac{2\pi kl}{N}\right)
$$

By reducing the sum to the range $k = 1, \dots, \frac{N}{2} - 1$, we get

$$X_l = \Re\{c_0\} + \Re\{c_{N/2}\} \cdot \cos(\pi l) + \sum_{k=1}^{\frac{N}{2}-1} \Re\{c_k\} \cdot \cos\left(\frac{2\pi kl}{N}\right) + \Re\{c_{-k}\} \cdot \cos\left(\frac{2\pi(-k)l}{N}\right) - \Im\{c_k\} \cdot \sin\left(\frac{2\pi kl}{N}\right) - \Im\{c_{-k}\} \cdot \sin\left(\frac{2\pi(-k)l}{N}\right)$$

Using the symmetry of the sine and cosine functions, this can be derived to

$$X_l = \Re\{c_0\} + \Re\{c_{N/2}\} \cdot \cos(\pi l) + \sum_{k=1}^{\frac{N}{2}-1} (\Re\{c_k\} + \Re\{c_{-k}\}) \cdot \cos\left(\frac{2\pi kl}{N}\right) + (\Im\{c_{-k}\} - \Im\{c_k\}) \cdot \sin\left(\frac{2\pi kl}{N}\right)$$

Since $c_{-k} = c_k^*$ it is $\Re\{c_{-k}\} = \Re\{c_k\}$ and $\Im\{c_{-k}\} = -\Im\{c_k\}$. So, we get

$$X_l = \Re\{c_0\} + \Re\{c_{N/2}\} \cdot \cos(\pi l) + \sum_{k=1}^{\frac{N}{2}-1} 2\Re\{c_k\} \cdot \cos\left(\frac{2\pi kl}{N}\right) - 2\Im\{c_k\} \cdot \sin\left(\frac{2\pi kl}{N}\right)$$

For $N = 12$, $a_k = 2\Re\{c_k\}, b_k = -2\Im\{c_k\}$ for all $k = 1, \dots, \frac{N}{2}$, $a_0 = c_0$ and $a_{\frac{N}{2}} = c_{\frac{N}{2}}$ we get equation (2):

$$X_l = a_0 + \sum_{k=1}^{5} \left(a_k \cos\left(\frac{\pi kl}{6}\right) + b_k \sin\left(\frac{\pi kl}{6}\right)\right) + a_6 \cos(\pi l)$$

Maple Demo

see Maple-Worksheet `pallas2.mws`.

Excercise 2

According to the Hint:

If $x_l = -X_{12-l}$, then $X_6 = -X_{12-6}$ holds. This is possible only if also $X_6 = 0$ holds. Analog we get $X_0 = -X_{12-0} = -X_{12}$. Here the value X_{12} is the declination for the ascension of

360°, which is the same as the declination value for 0° due to the periodicity of the data. Thus, it must also apply that $X_0 = X_{12}$. So we can conclude that $X_0 = 0$ must hold.

With these considerations we can compute the values a_k and b_k, for example with the Maple-Worksheet `pallas1.mws`.

According to the coefficients:

We guess that the interpolation of the axis symmetrical data should only need the axis symmetrical basis functions (i.e. all cos functions), while the interpolation of the point symmetrical data will only depends on the point symmetric basis functions (i.e. all sin functions). Hence, in one case all coefficients $b_k = 0$, while in the other all $a_k = 0$.

To show this we can insert the symmetrical constraints into the equation for the interpolation

$$X_l = a_0 + \sum_{k=1}^{5} \left(a_k \cos\left(\frac{\pi k l}{6}\right) + b_k \sin\left(\frac{\pi k l}{6}\right) \right) + a_6 \cos(\pi l) \,. \qquad (1)$$

For X_{12-l} it must hold:

$$\begin{aligned} X_{12-l} &= a_0 + \sum_{k=1}^{5} \left(a_k \cos\left(\frac{\pi k(12-l)}{6}\right) + b_k \sin\left(\frac{\pi k(12-l)}{6}\right) \right) + a_6 \cos(\pi(12-l)) \\ &= a_0 + \sum_{k=1}^{5} \left(a_k \cos\left(2\pi k - \frac{\pi k l}{6}\right) + b_k \sin\left(2\pi k - \frac{\pi k l}{6}\right) \right) + a_6 \cos(12\pi - l\pi) \\ &= a_0 + \sum_{k=1}^{5} \left(a_k \cos\left(\frac{\pi k l}{6}\right) - b_k \sin\left(\frac{\pi k l}{6}\right) \right) + a_6 \cos(\pi l) \end{aligned}$$

If $X_l = X_{12-l}$, then holds $X_l - X_{12-l} = 0$, i.e.:

$$\begin{aligned} & a_0 + \sum_{k=1}^{5} \left(a_k \cos\left(\frac{\pi k l}{6}\right) + b_k \sin\left(\frac{\pi k l}{6}\right) \right) + a_6 \cos(\pi l) \\ - & \left(a_0 + \sum_{k=1}^{5} \left(a_k \cos\left(\frac{\pi k l}{6}\right) - b_k \sin\left(\frac{\pi k l}{6}\right) \right) + a_6 \cos(\pi l) \right) = 0. \end{aligned}$$

Simplified this results in (for all l):

$$\begin{aligned} & 2 \sum_{k=1}^{5} b_k \sin\left(\frac{\pi k l}{6}\right) = 0 \\ \Rightarrow \quad & b_k = 0 \quad \text{for all } k. \end{aligned}$$

Remark: To be precise we would need to show that the solution $b_k = 0$ for all k is the only solution (uniqueness). We will skip this step here and note further that we would generally need to show the uniqueness of the trigonometric interpolation, since otherwise the excercise would not have been properly stated at all. The uniqueness follows from the fact that the matrix is invertible (See the matrix from the lecture, which is, however, for the complex DFT).

Analog we get, if $X_l = -X_{12-l}$ which means $X_l + X_{12-l} = 0$:

$$2a_0 + 2\sum_{k=1}^{5} a_k \cos\left(\frac{\pi k l}{6}\right) + 2a_6 \cos(\pi l) = 0$$

$$\Rightarrow \quad a_k = 0 \quad \text{for all } k.$$

Excercise 3: DFT and „Padding“

For the classic Fast Fourier Transform the number of discrete data must be a power of two. If this is not the case, one could try to fill up the dataset by ”zero” entries like this:

$$\hat{f}_n := \begin{cases} f_n & \text{if} \quad n \leq N-1 \\ 0 & \text{if} \quad N \leq n \leq M-1 \end{cases}$$

The Fourier coefficients $\widehat{F}_k$ of the extended dataset then add up to

$$\widehat{F}_k = \frac{1}{M}\sum_{n=0}^{M-1} \hat{f}_n \omega_M^{-kn} = \frac{1}{M}\sum_{n=0}^{N-1} f_n \omega_M^{-kn}.$$

This looks like if the $\widehat{F}_k$ are just the $\frac{N}{M}$ multiple of the original coefficients from the transform of length N:

$$F_k = \frac{1}{N}\sum_{n=0}^{N-1} f_n \omega_N^{-kn}.$$

However, this is not the case, since

$$\omega_N^{-kn} \neq \omega_M^{-kn}.$$

So, the frequencies of the base functions do change.

If we take the Fourier transform as an interpolation problem, then the extention of the dataset is equal to an increment of the number of supporting points. Since the observed interval stays the same ($[0, 2\pi]$), the distance between the supporting points must decrease. By padding the dataset with "zeros" we actually compressed the signal and therefore the signal must be assembled from higher-frequency oscillations.

We go on with the equation from above. First we show that

$$\omega_M^{-kn} = e^{-i2\pi kn/M} = e^{-i2\pi kn(N/M)/N} = \left(\omega_N^{-kn}\right)^{N/M}$$

holds and therefore

$$\widehat{F}_k = \frac{1}{M}\sum_{n=0}^{N-1} f_n \left(\omega_N^{-kn}\right)^{N/M} = \frac{N}{M}\cdot\frac{1}{N}\sum_{n=0}^{N-1} f_n \omega_N^{-k(N/M)n}$$

In general we cannot express this by the F_k. But if kN/M is an integer number, we get

$$\widehat{F}_k = \frac{N}{M}\cdot\frac{1}{N}\sum_{n=0}^{N-1} f_n \omega_N^{-k(N/M)n} = \frac{N}{M} F_{kN/M}$$

Explanation: The Fourier components $\widehat{F}_k$ of the compressed signal belong to the wavenumber k. In the original signal the same component would belong to the oscillation with wavenumber kN/M. If kN/M is an integer number, this Fourier component is also computed in the "short" transformation and can be taken from the "long" transformation directly without being changed. If kN/M is not an integer number, then there is no according component in the "short" transformation.